Logic Continued

Emil Straschil

ETH Zurich

2025

Propositional Logic

The **syntax** of propositional logic are as we know them from chapter 2.

Atomic Formula

An **atomic formula** is a symbol A_i.

We can combine atomic formulas using $\vee, \wedge, \neg$ to build larger formulas.

Propositional Logic

The **syntax** of propositional logic are as we know them from chapter 2.

Atomic Formula

An **atomic formula** is a symbol A_i.

We can combine atomic formulas using $\vee, \wedge, \neg$ to build larger formulas.

Interpretation

An **interpretation** $\mathcal{A}$ in prop. logic is **suitable** for a formula F if it assigns each atomic formula in F a truth value.

Interpretation Example

Take the formula

$$F = (A_1 \vee A_2) \wedge A_3$$

and the interpretation $\mathcal{A}$ over $Z = \{A_1, A_2, A_3\}$ with

$$\mathcal{A}(A_1) = 0, \mathcal{A}(A_2) = 1, \mathcal{A}(A_3) = 1$$

$\mathcal{A}$ is suitable for F. And it is even a **model**, since the formula evaluates to true under the truth assignment.

Interpretation Example

Take the formula

$$F = (A_1 \vee A_2) \wedge A_3$$

and the interpretation $\mathcal{B}$ over $Z = \{A_1, A_3\}$ with

$$\mathcal{B}(A_1) = 1, \mathcal{B}(A_3) = 1$$

$\mathcal{B}$ is **not** suitable for F since it does not assign a value to A_2.

Note that even though for any truth assignment we could add for A_2 to $\mathcal{B}$, F would evaluate to true, it is still not suitable!

Predicate Logic - Syntax

1. **Variable symbols** are of the form x_i. (We often use x, y, z.)
2. $f_i^{(k)}$ is a **function** with k arguments. (We often omit the argument count and just write f, g, h.)
3. A **constant** is a function with no arguments.
4. $P_i^{(k)}$ is a **predicate** symbol with k arguments. (We just write P, Q, R.)
5. A **term** can be built with variables and functions.

Predicate Logic - Formulas

1. For some terms $t_1, \ldots, t_k$ an **atomic formula** is $P_i^{(k)}(t_1, \ldots, t_k)$.
2. For formulas F and G, $F \wedge G, F \vee G, \neg F$ are formulas.
3. For a formula F, $\forall x_i F$ and $\exists x_i F$ are formulas

Syntax Examples

x_1 is a term and $f_1(x_1)$ is a term. Now if we have a function $f_2^{(2)}$ we can combine them to

$$f_2(x_1, f_1(x_1)).$$

$P(x_1)$ is a formula and $P(f_1(x_1))$ is too. We can combine them to

$$P(x_1) \wedge P(f_1(x_1)).$$

$P(x_2)$ is a formula too. And so is

$$\forall x_1 P(x_2).$$

Note that there is no reason why x should appear in the formula F for $\forall x F$.

Free Variables

Free Variable

A *free variable* not bound by a quantifier. All free variables have to be defined by an interpretation of the formula.

For example, in the formula $\forall x P(x, y)$, the x is bound by the quantifier, but y is free.

Substitution

$F[x/t]$ it the formula F, but we substitute every free occurrence of x with t.

For example, $P(x)[x/f(y)] = P(f(y))$.

Predicate Logic - Interpretation

An **interpretation** needs to define:

1. The universe U we are working in
2. All function symbols.
3. All predicate symbols
4. All *free* variables

All the functions defined go from U to U and all the values assigned to free variables are in U.
If an interpretation $\mathcal{A}$ defines all this for a formula F, then it is **suitable** for that formula.
We write $U^{\mathcal{A}}, x^{\mathcal{A}}, f^{\mathcal{A}}, P^{\mathcal{A}}$ for the stuff defined by $\mathcal{A}$.

Interpretation Example

Say we have the formula

$$F = \forall x(P(x) \vee P(f(x, y))).$$

A suitable interpretation should define U, P, f, y. Say we take $\mathcal{A}$ with:

$$\begin{aligned} U^{\mathcal{A}} &= \mathbb{N} \\ P^{\mathcal{A}}(x) &= even(x) \\ f^{\mathcal{A}}(x, y) &= x + y \\ y^{\mathcal{A}} &= 1 \end{aligned}$$

Then it reads "for all natural numbers x, either x or $x + 1$ is even". And $\mathcal{A}$ is a model for F.

Important Info

In prop. logic we interpreted the variables $A, B, \ldots$ with **truth values**. In predicate logic we interpret the free variables with **values from the universe**. So something like this:

$$P(x_1) \vee x_2$$

is **not** a valid formula.

(We can however achieve something similar by using predicates, for example of the form $P_i^{(0)}$.)

Predicate Logic - Semantics

For some formula F and interpretation $\mathcal{A}$ we want to find the truth value $\mathcal{A}(F)$.

- For terms, functions, variables we simply evaluate them as defined
- We also evaluate predicates as defined by $\mathcal{A}$

Overwriting

$\mathcal{A}_{[x \to u]}$ is the same as $\mathcal{A}$, but we overwrite x with u.

- $\mathcal{A}(\forall x F) = 1$ if $\mathcal{A}_{[x \to u]}(F) = 1$ for all u in U
- $\mathcal{A}(\exists x F) = 1$ if $\mathcal{A}_{[x \to u]} = 1$ for some u in U

Useful Properties

You can find some useful properties in Lemma 6.7 in the script. Note that for a lot of exercises you are not allowed to use those (if explicitly stated).

Semantics Example

We show that $\forall x P(x) \models \neg \exists x (\neg P(f(x)))$.

Semantics Example

We show that $\forall x P(x) \models \neg \exists x (\neg P(f(x)))$.
Intuition: If P(x) holds for all values in the universe, then there cannot be any value that f(x) maps to, for which P(x) does not hold. So let's do this formally:

(More) Formally

We show that $\forall x P(x) \models \neg \exists x (\neg P(f(x)))$.

(More) Formally

We show that $\forall x P(x) \models \neg\exists x(\neg P(f(x)))$. Take any interpretation $\mathcal{A}$ with $\mathcal{A}(\forall x P(x)) = 1$.

$$\iff \mathcal{A}_{[x \to u]} P(x) = 1 \text{ for all } u \text{ in } U \tag{1}$$

$$\implies \mathcal{A}_{[x \to f(u)]} P(x) = 1 \text{ for all } u \text{ in } U \tag{2}$$

(More) Formally

We show that $\forall x P(x) \models \neg\exists x(\neg P(f(x)))$. Take any interpretation $\mathcal{A}$ with $\mathcal{A}(\forall x P(x)) = 1$.

$$\begin{aligned} &\iff \mathcal{A}_{[x \to u]} P(x) = 1 \text{ for all } u \text{ in } U && (1) \\ &\implies \mathcal{A}_{[x \to f(u)]} P(x) = 1 \text{ for all } u \text{ in } U && (2) \end{aligned}$$

What needs to hold for $\mathcal{A}(\neg\exists x(\neg P(f(x)))) = 1$?

$$\begin{aligned} &\iff \mathcal{A}(\exists x(\neg P(f(x)))) = 0 && (3) \\ &\iff \text{not } \mathcal{A}(\exists x(\neg P(f(x)))) = 1 && (4) \\ &\iff \text{not } \mathcal{A}_{[x \to u]}(\neg P(f(x))) = 1 \text{ for some } u \text{ in } U && (5) \\ &\iff \mathcal{A}_{[x \to u]}(\neg P(f(x))) = 0 \text{ for all } u \text{ in } U && (6) \\ &\iff \mathcal{A}_{[x \to u]} P(f(x)) = 1 \text{ for all } u \text{ in } U && (7) \\ &\iff \mathcal{A}_{[x \to f(u)]} P(x) = 1 \text{ for all } u \text{ in } U && (8) \end{aligned}$$

(More) Formally

We show that $\forall x P(x) \models \neg\exists x(\neg P(f(x)))$. Take any interpretation $\mathcal{A}$ with $\mathcal{A}(\forall x P(x)) = 1$.

$$\iff \mathcal{A}_{[x \to u]} P(x) = 1 \text{ for all } u \text{ in } U \quad (1)$$
$$\implies \mathcal{A}_{[x \to f(u)]} P(x) = 1 \text{ for all } u \text{ in } U \quad (2)$$

What needs to hold for $\mathcal{A}(\neg\exists x(\neg P(f(x)))) = 1$?

$$\iff \mathcal{A}(\exists x(\neg P(f(x)))) = 0 \quad (3)$$
$$\iff \text{not } \mathcal{A}(\exists x(\neg P(f(x)))) = 1 \quad (4)$$
$$\iff \text{not } \mathcal{A}_{[x \to u]}(\neg P(f(x))) = 1 \text{ for some } u \text{ in } U \quad (5)$$
$$\iff \mathcal{A}_{[x \to u]}(\neg P(f(x))) = 0 \text{ for all } u \text{ in } U \quad (6)$$
$$\iff \mathcal{A}_{[x \to u]} P(f(x)) = 1 \text{ for all } u \text{ in } U \quad (7)$$
$$\iff \mathcal{A}_{[x \to f(u)]} P(x) = 1 \text{ for all } u \text{ in } U \quad (8)$$

We see that the LHS $\implies$ (2) $\implies$ (8) $\implies$ RHS.

Justifications

(1) semantics of $\forall$
(2) $f(u)$ is an element of the universe U
(3) semantics of $\neg$
(4) $\mathcal{A}(F)$ is either 0 or 1. So it is exactly 0 when it is not 1
(5) semantics of $\exists$
(6) if something does not hold for any (some) u in U, then the opposite must hold for all u in U
(7) semantics of $\neg$
(8) replacing x with u in $f(x)$ is the same as replacing x with $f(u)$ in x.

How to Prove Stuff Like This

Most proofs work the same way:

1. Before you start your proof, try to understand the meaning of the formulas inuitively
2. Apply definitions until you get to language.
3. Reason with language
4. Often times simplifying both sides separately is easier than doing one giant chain of implications

Be careful when negating quantifiers and work with "not" to be on the safe side

Don't forget to justify each step and make a clear argument when you reason with natural language.

Exercise 1

Prove or disprove each of the following statements. Do not use any theorems or lemmas from the lecture notes. Note that x may appear free in F, G or both.

a) For any formulas F and G, we have

$$\forall x\ (F \wedge G)\ \models\ (\forall x\ F) \wedge G.$$

b) For any formulas F and G, we have

$$\exists x\ (F \wedge G)\ \models\ (\exists x\ F) \wedge G.$$

Solution 1a

a) The statement is true.

Proof. Let $\mathcal{A}$ be any interpretation suitable for both $\forall x\,(F \wedge G)$ and $(\forall x\, F) \wedge G$, such that $\mathcal{A}\,(\forall x\,(F \wedge G)) = 1$. According to the semantics of $\forall$, we have $\mathcal{A}_{[x \to u]}(F \wedge G) = 1$ for all $u \in U$. According to the semantics of $\wedge$, we further have (1) $\mathcal{A}_{[x \to u]}(F) = 1$ for all $u \in U$ and (2) $\mathcal{A}_{[x \to u]}(G) = 1$ for all $u \in U$.

The fact (1) implies (3) $\mathcal{A}(\forall x\, F) = 1$, according to the semantics of $\forall$. Furthermore, note that if x appears free in G, then it also appears free in $(\forall x\, F) \wedge G$, and since $\mathcal{A}$ is suitable for $(\forall x\, F) \wedge G$, it must assign a value to x. We now define u^* as follows: if x appears free in G, then u^* is the value assigned to x by $\mathcal{A}$, else u^* is arbitrary. By the definition of u^*, we have $\mathcal{A}_{[x \to u^*]}(G) = \mathcal{A}(G)$, so by (2), we have (4) $\mathcal{A}(G) = 1$.

The facts (3) and (4) imply that $\mathcal{A}\,((\forall x\, F) \wedge G) = 1$.

Solution 1b

b) The statement is false.

Counterexample. Let $F = P(x)$ and $G = Q(x)$. Let $\mathcal{A}$ be the interpretation with the universe $U^{\mathcal{A}} = \{0, 1\}$, which defines:

- $P^{\mathcal{A}}(0) = 1$ and $P^{\mathcal{A}}(1) = 1$
- $Q^{\mathcal{A}}(0) = 1$ and $Q^{\mathcal{A}}(1) = 0$
- $x^{\mathcal{A}} = 1$

We then have $\mathcal{A}(\exists x\ (P(x) \wedge Q(x))) = 1$, because $\mathcal{A}_{[x \to 0]}(P(x) \wedge Q(x)) = 1$. However, $\mathcal{A}((\exists x\ P(x)) \wedge Q(x)) = 0$, because $\mathcal{A}(Q(x)) = 0$.

Exercise 2

Prove the following statements using the semantics of predicate logic (Definition 6.36). Do not use other results from the lecture notes.

a) $\exists x \, \forall y P(x, y) \models \exists x \, P(x, f(x))$.

b) $\neg(\forall x \, P(x)) \models \exists x \, \neg P(x)$.

Solution 2a

a) Let $\mathcal{A}$ be an interpretation which is suitable both for $\exists x\, \forall y\, P(x,y)$ and for $\exists x P(x, f(x))$, and such that $\mathcal{A}$ is a model for $\exists x\, \forall y\, P(x,y)$, meaning $\mathcal{A}(\exists x\, \forall y\, P(x,y)) = 1$. In particular, observe that $f^{\mathcal{A}} : U^{\mathcal{A}} \to U^{\mathcal{A}}$ is defined. We have

$$\begin{aligned}
&\mathcal{A}(\exists x\, \forall y\, P(x,y)) = 1 && \\
\overset{\cdot}{\iff} &\mathcal{A}_{[x \to u]}(\forall y\, P(x,y)) = 1 \text{ for some } u \in U^{\mathcal{A}} && (\text{Semantics of } \exists) \\
\overset{\cdot}{\iff} &\mathcal{A}_{[x \to u][y \to v]}(P(x,y)) = 1 \text{ for some } u \in U^{\mathcal{A}} \text{ and all } v \in U^{\mathcal{A}} && (\text{Semantics of } \forall) \\
\overset{\cdot}{\implies} &\mathcal{A}_{[x \to u][y \to f^{\mathcal{A}}(u)]}(P(x,y)) = 1 \text{ for some } u \in U^{\mathcal{A}} && (f^{\mathcal{A}}(u) \in U^{\mathcal{A}}) \\
\overset{\cdot}{\iff} &\mathcal{A}_{[x \to u]}(P(x, f(x))) = 1 \text{ for some } u \in U^{\mathcal{A}} && (\text{Semantics of } f(x)) \\
\overset{\cdot}{\iff} &\mathcal{A}(\exists x\, P(x, f(x))) = 1. && (\text{Semantics of } \exists)
\end{aligned} \tag{1}$$

Solution 2b

b) Let $\mathcal{A}$ be an interpretation which is suitable both for $\neg(\forall x\, P(x))$ and for $\exists x\, \neg P(x)$, and such that $\mathcal{A}$ is a model for $\neg(\forall x\, P(x))$, meaning $\mathcal{A}(\neg(\forall x\, P(x))) = 1$. We have

$$
\begin{aligned}
& \mathcal{A}(\neg(\forall x\, P(x))) = 1 & \\
\dot{\iff} & \mathcal{A}(\forall x\, P(x)) = 0 & \text{(Semantics of } \neg) \\
\dot{\iff} & \mathcal{A}_{[x \to u]}(P(x)) = 0 \text{ for some } u \in U^{\mathcal{A}} & (*) \\
\dot{\iff} & \mathcal{A}_{[x \to u]}(\neg P(x)) = 1 \text{ for some } u \in U^{\mathcal{A}} & \text{(Semantics of } \neg) \\
\dot{\iff} & \mathcal{A}(\exists x\, \neg P(x)) = 1. & \text{(Semantics of } \exists)
\end{aligned}
\tag{2}
$$

- $(*)$ If for all assignments of the variable x it holds that $\mathcal{A}_{[x \to u]}(P(x)) = 1$, then by the semantics of $\forall$ we get $\mathcal{A}(\forall x\, P(x))) = 1$, a contradiction.