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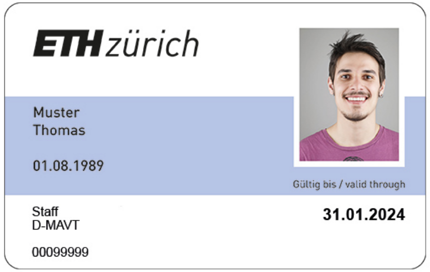
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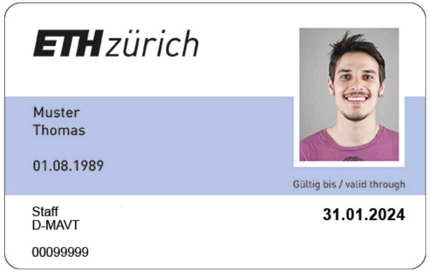
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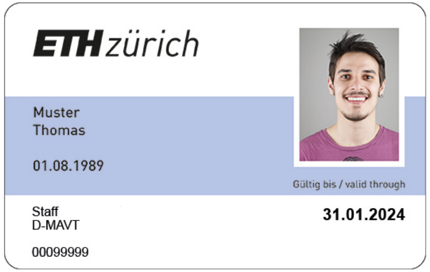
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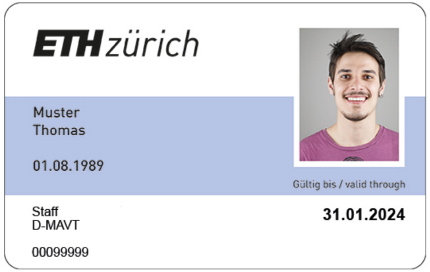
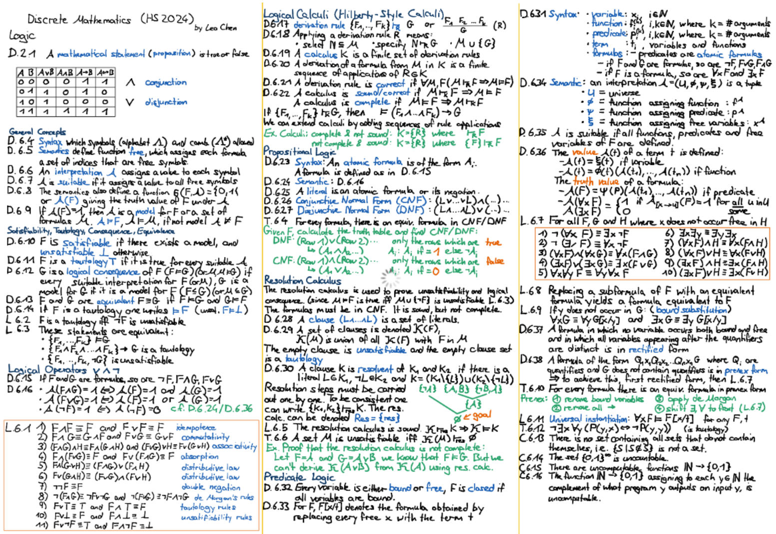




Discrete Mathematics (HS 2024) by Leo Chen

Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	$A\vee B$	$A\wedge B$	$A\to B$	$A\leftrightarrow B$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

$\wedge$ conjunction
$\vee$ disjunction

General Concepts

D.6.4 Syntax which symbols (alphabet Λ) and comb. (Λ^*) allowed
D.6.5 Semantics define function free, which assigns each formula a set of indices that are free symbols.
D.6.6 An interpretation $\mathcal{A}$ assigns a value to each symbol
D.6.7 $\mathcal{A}$ is suitable if it assigns a value to all free symbols
D.6.8 The semantics also define a function $\sigma(F,\mathcal{A})=\{0,1\}$ or $\mathcal{A}(F)$ giving the truth value of F under $\mathcal{A}$
D.6.9 If $\mathcal{A}(F)=1$, then $\mathcal{A}$ is a model for F or a set of formulas M. $\mathcal{A}\models F$, $\mathcal{A}\models M$, if not model $\mathcal{A}\not\models F$

Satisfiability, Tautology, Consequence, Equivalence

D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise
D.6.11 F is a tautology $\top$ if it is true for every suitable $\mathcal{A}$
D.6.12 G is a logical consequence of F ($F\models G$) (or M, $M\models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F\leq G$) (or $M\leq G$)
D.6.13 F and G are equivalent $F\equiv G$ if $F\models G$ and $G\models F$
D.6.14 If F is a tautology one writes $\models F$ (unsat. $F\models\perp$)
L.6.2 F is a tautology iff $\neg F$ is unsatisfiable
L.6.3 These statements are equivalent:
- $\{F_1,\dots,F_k\}\models G$
- $\{F_1\wedge F_2\wedge\dots\wedge F_k\}\to G$ is a tautology
- $\{F_1,\dots,F_k,\neg G\}$ is unsatisfiable

Logical Operators $\vee\wedge\neg$

D.6.15 If F and G are formulas, so are $\neg F$, $F\wedge G$, $F\vee G$
D.6.16
- $\mathcal{A}(F\wedge G)=1 \iff \mathcal{A}(F)=1$ and $\mathcal{A}(G)=1$
- $\mathcal{A}(F\vee G)=1 \iff \mathcal{A}(F)=1$ or $\mathcal{A}(G)=1$
- $\mathcal{A}(\neg F)=1 \iff \mathcal{A}(\neg F)=0$ cf. D.6.24/D.6.36

L.6.1
1) $F\wedge F\equiv F$ and $F\vee F\equiv F$ idempotence
2) $F\wedge G\equiv G\wedge F$ and $F\vee G\equiv G\vee F$ commutativity
3) $(F\wedge G)\wedge H\equiv F\wedge(G\wedge H)$ and $(F\vee G)\vee H\equiv F\vee(G\vee H)$ associativity
4) $F\wedge(F\vee G)\equiv F$ and $F\vee(F\wedge G)\equiv F$ absorption
5) $F\wedge(G\vee H)\equiv(F\wedge G)\vee(F\wedge H)$ distributive law
6) $F\vee(G\wedge H)\equiv(F\vee G)\wedge(F\vee H)$ distributive law
7) $\neg\neg F\equiv F$ double negation
8) $\neg(F\wedge G)\equiv\neg F\vee\neg G$ and $\neg(F\vee G)\equiv\neg F\wedge\neg G$ de Morgan's rules
9) $F\vee\top\equiv\top$ and $F\wedge\top\equiv F$ tautology rules
10) $F\vee\perp\equiv F$ and $F\wedge\perp\equiv\perp$ unsatisfiability rules
11) $F\vee\neg F\equiv\top$ and $F\wedge\neg F\equiv\perp$

Logical Calculi (Hilberty-Style Calculi)

D.6.17 derivation rule $\{F_1,\dots,F_k\}\vdash_R G$ or $\dfrac{F_1\ F_2\ \dots\ F_k}{G}$ (R)
D.6.18 Applying a derivation rule R means:
- select $N\subseteq M$ · specify $N\vdash_R G$ · $M\cup\{G\}$

D.6.19 A calculus K is a finite set of derivation rules
D.6.20 A derivation of a formula from M in K is a finite sequence of applications of $R\in K$
D.6.21 A derivation rule is correct if $\forall M,F\,(M\vdash_R F\Rightarrow M\models F)$
D.6.22 A calculus is sound/correct if $M\vdash_K F\Rightarrow M\models F$
A calculus is complete if $M\models F\Rightarrow M\vdash_K F$
If $\{F_1,\dots,F_k\}\vdash_K G$, then $\models(F_1\wedge\dots\wedge F_k)\to G$
We can extend calculi by adding sequences of rule applications
Ex. Calculi: complete & not sound: $K=\{R\}$ where $\vdash_R F$
not complete & sound: $K=\{R\}$ where $\{F\}\vdash_R F$

Propositional Logic

D.6.23 Syntax: An atomic formula is of the form A_i. A formula is defined as in D.6.15
D.6.24 Semantic: D.6.16
D.6.25 A literal is an atomic formula or its negation.
D.6.26 Conjunctive Normal Form (CNF): $(L\vee\dots\vee L)\wedge(\dots)\dots$
D.6.27 Disjunctive Normal Form (DNF): $(L\wedge\dots\wedge L)\vee(\dots)\dots$
T.6.4 For every formula, there is an equiv. formula in CNF/DNF
Given F, calculate the truth table and find CNF/DNF:
DNF: (Row 1)$\vee$(Row 2)… only the rows which are true
↳ $(A_1\wedge A_2\dots)$ A_i: A_i if $=1$ else $\neg A_i$
CNF: (Row 1)$\wedge$(Row 2)… only the rows which are false
↳ $(A_1\vee A_2\dots)$ A_i: A_i if $=0$ else $\neg A_i$

Resolution Calculus

The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M\models F$ is true iff $M\cup\{\neg F\}$ is unsatisfiable L.6.3) The formulas must be in CNF. It is sound, but not complete.
D.6.28 A clause $(L\wedge\dots\wedge L)$ is a set of literals.
D.6.29 A set of clauses is denoted $\mathcal{K}(F)$, $\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M
The empty clause is unsatisfiable and the empty clause set is a tautology
D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L\in K_1$, $\neg L\in K_2$ and $K=(K_1\setminus\{L\})\cup(K_2\setminus\{\neg L\})$
Resolution steps must be carried out one by one. To be consistent one can write $\{K_1,K_2\}\vdash_{res}K$. The res. calc. can be denoted $Res=\{res\}$

$\{\neg A\}$ $\{A,B\}$ $\{\neg B,A\}$
$\{A\}$
$\emptyset$ ← goal

L.6.5 The resolution calculus is sound. $\mathcal{K}\vdash_{res}K\Rightarrow\mathcal{K}\models K$
T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M)\vdash_{res}\emptyset$
Ex. Proof that the resolution calculus is not complete:
Let $F=A$ and $G=A\vee B$ we know that $F\models G$. But we can't derive $\mathcal{K}(A\vee B)$ from $\mathcal{K}(A)$ using res. calc.

Predicate Logic

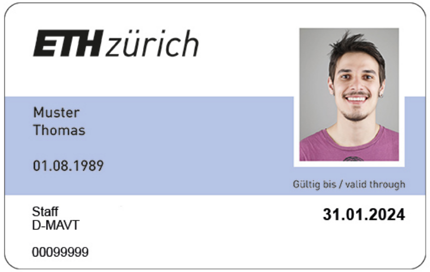
D.6.32 Every variable is either bound or free, F is closed if all variables are bound.
D.6.33 For F, $F[x/t]$ denotes the formula obtained by replacing every free x with the term t
D.6.31 Syntax:
- variable: x_i, $i\in\mathbb{N}$
- function: $f_i^{(k)}$, $i,k\in\mathbb{N}$, where k = #arguments
- predicate: $P_i^{(k)}$, $i,k\in\mathbb{N}$, where k = #arguments
- term: t_i, variables and functions
- formulas: – predicates are atomic formulas
  – if F and G are formulas, so are $\neg F$, $F\vee G$, $F\wedge G$
  – if F is a formula, so are $\forall xF$ and $\exists xF$

D.6.34 Semantic: an interpretation $\mathcal{A}=(U,\phi,\psi,\xi)$ is a tuple
- U = universe
- ϕ = function assigning function: $f^{\mathcal{A}}$
- ψ = function assigning predicate: $P^{\mathcal{A}}$
- ξ = function assigning free variables: $x^{\mathcal{A}}$

D.6.35 $\mathcal{A}$ is suitable if all functions, predicates and free variables of F are defined.
D.6.36 The value $\mathcal{A}(t)$ of a term t is defined:
– $\mathcal{A}(t)=\xi(t)$ if variable
– $\mathcal{A}(t)=\phi(t)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if function
The truth value of a formula:
– $\mathcal{A}(F)=\psi(P)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if predicate
– $\mathcal{A}(\forall xF)=\begin{cases}1 & \text{if } \mathcal{A}_{[x\to u]}(F)=1 \text{ for all } u \text{ in } U\\ 0\end{cases}$
$\mathcal{A}(\exists xF)=\begin{cases}1 & \text{some}\\ 0\end{cases}$

L.6.7 For all F, G and H where x does not occur free in H
1) $\neg(\forall xF)\equiv\exists x\neg F$
2) $\neg(\exists xF)\equiv\forall x\neg F$
3) $(\forall xF)\wedge(\forall xG)\equiv\forall x(F\wedge G)$
4) $(\exists xF)\vee(\exists xG)\equiv\exists x(F\vee G)$
5) $\forall x\forall yF\equiv\forall y\forall xF$
6) $\exists x\exists y\equiv\exists y\exists x$
7) $(\forall xF)\wedge H\equiv\forall x(F\wedge H)$
8) $(\forall xF)\vee H\equiv\forall x(F\vee H)$
9) $(\exists xF)\wedge H\equiv\exists x(F\wedge H)$
10) $(\exists xF)\vee H\equiv\exists x(F\vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F
L.6.9 If y does not occur in G (bound substitution) $\forall xG\equiv\forall yG[x/y]$ and $\exists xG\equiv\exists yG[x/y]$
D.6.37 A formula in which no variable occurs both bound and free and in which all variables appearing after the quantifiers are distinct is in rectified form
D.6.38 A formula of the form $Q_1x_1Q_2x_2\dots Q_nx_nG$ where Q_i are quantifiers and G does not contain quantifiers is in prenex form ⇒ to achieve this, first rectified form, then L.6.7
T.6.10 For every formula there is an equiv. formula in prenex form
Prenex: ① rename bound variables ② apply de Morgan ③ remove all → ④ shift $\exists,\forall$ to front (L.6.7)
L.6.11 Universal instantiation: $\forall xF\models F[x/t]$ for any F, t
T.6.12 $\neg\exists x\forall y\,(P(y,x)\leftrightarrow\neg P(y,y))$ (is tautology)
C.6.13 There is no set containing all sets that do not contain themselves, i.e. $\{S\mid S\notin S\}$ is not a set.
C.6.14 The set $\{0,1\}^\infty$ is uncountable.
C.6.15 There are uncomputable functions $\mathbb{N}\to\{0,1\}$
C.6.16 The function $\mathbb{N}\to\{0,1\}$ assigning to each $y\in\mathbb{N}$ the complement of what program y outputs on input y, is uncomputable.



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- $\mathcal{A}(\neg F)=1 \iff \mathcal{A}(\neg F)=0$ cf. D.6.24/D.6.36

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9) $F\vee T\equiv T$ and $F\wedge T\equiv F$ tautology rules
10) $F\vee\perp\equiv F$ and $F\wedge\perp\equiv\perp$ unsatifiability rules
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Logical Calculi (Hilberty-Style Calculi)

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If $\{F_1,\dots,F_k\}\vdash_K G$, then $\models(F_1\wedge\dots\wedge F_k)\to G$

We can extend calculi by adding sequences of rule applications

Ex. Calculi: complete & not sound: $K=\{R\}$ where $\vdash_R F$
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Resolution steps must be carried out one by one. To be consistent one can write $\{K_1,K_2\}\vdash_{res}K$. The res. calc. can be denoted $Res=\{res\}$

$\{\neg A\}$ $\{A,B\}$ $\{\neg B,A\}$
$\{A\}$
$\emptyset$ ← goal

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Ex. Proof that the resolution calculus is not complete:
Let $F=A$ and $G=A\vee B$ we know that $F\models G$. But we can't derive $\mathcal{K}(A\vee B)$ from $\mathcal{K}(A)$ using res. calc.

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- term: t_i, variables and functions
- formulas: – predicates are atomic formulas
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– $\mathcal{A}(t)=\xi(t)$ if variable
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The truth value of a formula:
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L.6.7 For all F, G and H where x does not occur free in H

1) $\neg(\forall x F)\equiv\exists x\neg F$	6) $\exists x\exists y\equiv\exists y\exists x$
2) $\neg(\exists x F)\equiv\forall x\neg F$	7) $(\forall x F)\wedge H\equiv\forall x(F\wedge H)$
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L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F

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D.6.38 A formula of the form $Q_1x_1Q_2x_2\dots Q_nx_nG$ where Q_i are quantifiers and G does not contain quantifiers is in prenex form
$\Rightarrow$ to achieve this, first rectified form, then L.6.7

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Prenex: ① rename bound variables ② remove all $\to$ ③ apply de Morgan ④ shift $\exists,\forall$ to front (L.6.7)

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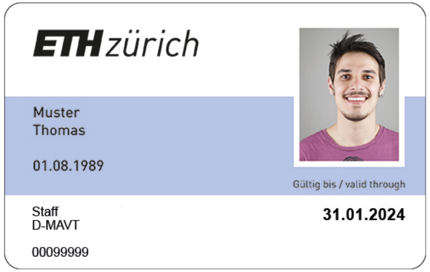

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D.6.9 If $\mathcal{A}(F) = 1$, then $\mathcal{A}$ is a model for F or a set of formulas M. $\mathcal{A} \models F$, $\mathcal{A} \models M$, if not model $\mathcal{A} \not\models F$

Satisfiability, Tautology, Consequence, Equivalence

D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise

D.6.11 F is a tautology $\top$ if it is true for every suitable $\mathcal{A}$

D.6.12 G is a logical consequence of F ($F \models G$) (or M, $M \models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F \leq G$) (or $M \leq G$)

D.6.13 F and G are equivalent $F \equiv G$ if $F \models G$ and $G \models F$

D.6.14 If F is a tautology one writes $\models F$ (unsat. $F \models \perp$)

L.6.2 F is a tautology iff $\neg F$ is unsatisfiable

L.6.3 These statements are equivalent:
- $\{F_1, \dots, F_k\} \models G$
- $\{F_1 \wedge F_2 \wedge \dots \wedge F_k\} \to G$ is a tautology
- $\{F_1, \dots, F_k, \neg G\}$ is unsatisfiable

Logical Operators $\vee \wedge \neg$

D.6.15 If F and G are formulas, so are $\neg F$, $F \wedge G$, $F \vee G$

D.6.16
- $\mathcal{A}(F \wedge G) = 1 \Leftrightarrow \mathcal{A}(F) = 1$ and $\mathcal{A}(G) = 1$
- $\mathcal{A}(F \vee G) = 1 \Leftrightarrow \mathcal{A}(F) = 1$ or $\mathcal{A}(G) = 1$
- $\mathcal{A}(\neg F) = 1 \Leftrightarrow \mathcal{A}(\neg F) = 0$ cf. D.6.24/D.6.36

L.6.1
1) $F \wedge F \equiv F$ and $F \vee F \equiv F$ idempotence
2) $F \wedge G \equiv G \wedge F$ and $F \vee G \equiv G \vee F$ commutativity
3) $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$ and $(F \vee G) \vee H \equiv F \vee (G \vee H)$ associativity
4) $F \wedge (F \vee G) \equiv F$ and $F \vee (F \wedge G) \equiv F$ absorption
5) $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ distributive law
6) $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$ distributive law
7) $\neg\neg F \equiv F$ double negation
8) $\neg(F \wedge G) \equiv \neg F \vee \neg G$ and $\neg(F \vee G) \equiv \neg F \wedge \neg G$ de Morgan's rules
9) $F \vee \top \equiv \top$ and $F \wedge \top \equiv F$ tautology rules
10) $F \vee \perp \equiv F$ and $F \wedge \perp \equiv \perp$ unsatifiability rules
11) $F \vee \neg F \equiv \top$ and $F \wedge \neg F \equiv \perp$

Logical Calculi (Hilberty-Style Calculi)

D.6.17 derivation rule $\{F_1, \dots, F_k\} \vdash_R G$ or $\dfrac{F_1 \; F_2 \; \dots \; F_k}{G}$ (R)

D.6.18 Applying a derivation rule R means:
- select $N \subseteq M$ • specify $N \vdash_R G$ • $M \cup \{G\}$

D.6.19 A calculus K is a finite set of derivation rules

D.6.20 A derivation of a formula from M in K is a finite sequence of applications of $R \in K$

D.6.21 A derivation rule is correct if $\forall M, F (M \vdash_R F \Rightarrow M \models F)$

D.6.22 A calculus is sound/correct if $M \vdash_K F \Rightarrow M \models F$
A calculus is complete if $M \models F \Rightarrow M \vdash_K F$

If $\{F_1, \dots, F_k\} \vdash_K G$, then $\models (F_1 \wedge \dots \wedge F_k) \to G$

We can extend calculi by adding sequences of rule applications

Ex. Calculi: complete & not sound: $K = \{R\}$ where $\vdash_R F$
not complete & sound: $K = \{R\}$ where $\{F\} \vdash_R F$

Propositional Logic

D.6.23 Syntax: An atomic formula is of the form A_i.
A formula is defined as in D.6.15

D.6.24 Semantic: D.6.16

D.6.25 A literal is an atomic formula or its negation.

D.6.26 Conjunctive Normal Form (CNF): $(L \vee \dots \vee L) \wedge (\dots) \dots$

D.6.27 Disjunctive Normal Form (DNF): $(L \wedge \dots \wedge L) \vee (\dots) \dots$

T.6.4 For every formula, there is an equiv. formula in CNF/DNF

Given F, calculate the truth table and find CNF/DNF:
DNF: (Row 1) $\vee$ (Row 2) … only the rows which are true
↳ $(A_1 \wedge A_2 \dots)$ A_i: A_i if $= 1$ else $\neg A_i$
CNF: (Row 1) $\wedge$ (Row 2) … only the rows which are false
↳ $(A_1 \vee A_2 \dots)$ A_i: A_i if $= 0$ else $\neg A_i$

Resolution Calculus

The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M \models F$ is true iff $M \cup \{\neg F\}$ is unsatisfiable L.6.3) The formulas must be in CNF. It is sound, but not complete.

D.6.28 A clause $(L \wedge \dots \wedge L)$ is a set of literals.

D.6.29 A set of clauses is denoted $\mathcal{K}(F)$,
$\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M

The empty clause is unsatisfiable and the empty clause set is a tautology

D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L \in K_1$, $\neg L \in K_2$ and $K = (K_1 \setminus \{L\}) \cup (K_2 \setminus \{\neg L\})$

Resolution steps must be carried out one by one. To be consistent one can write $\{K_1, K_2\} \vdash_{res} K$. The res. calc. can be denoted $Res = \{res\}$

$\{\neg A\}$ $\{A, B\}$ $\{\neg B, A\}$
$\{A\}$
$\emptyset$ ← goal

L.6.5 The resolution calculus is sound. $\mathcal{K} \vdash_{res} K \Rightarrow \mathcal{K} \models K$

T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M) \vdash_{res} \emptyset$

Ex. Proof that the resolution calculus is not complete:
Let $F = A$ and $G = A \vee B$ we know that $F \models G$. But we can't derive $\mathcal{K}(A \vee B)$ from $\mathcal{K}(A)$ using res. calc.

Predicate Logic

D.6.32 Every variable is either bound or free, F is closed if all variables are bound.

D.6.33 For F, $F[x/t]$ denotes the formula obtained by replacing every free x with the term t

D.6.31 Syntax:
- variable: x_i, $i \in \mathbb{N}$
- function: $f_i^{(k)}$, $i, k \in \mathbb{N}$, where k = # arguments
- predicate: $P_i^{(k)}$, $i, k \in \mathbb{N}$, where k = # arguments
- term: t_i, variables and functions
- formulas: – predicates are atomic formulas
  – if F and G are formulas, so are $\neg F$, $F \vee G$, $F \wedge G$
  – if F is a formula, so are $\forall x F$ and $\exists x F$

D.6.34 Semantic: an interpretation $\mathcal{A} = (U, \phi, \psi, \xi)$ is a tuple
- U = universe
- ϕ = function assigning function : $f^{\mathcal{A}}$
- ψ = function assigning predicate : $p^{\mathcal{A}}$
- ξ = function assigning free variables : $x^{\mathcal{A}}$

D.6.35 $\mathcal{A}$ is suitable if all functions, predicates and free variables of F are defined.

D.6.36 The value $\mathcal{A}(t)$ of a term t is defined:
– $\mathcal{A}(t) = \xi(t)$ if variable
– $\mathcal{A}(t) = \phi(t)(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))$ if function
The truth value of a formula:
– $\mathcal{A}(F) = \psi(P)(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))$ if predicate
– $\mathcal{A}(\forall x F)$, $\mathcal{A}(\exists x F)$ $= \begin{cases} 1 & \text{if } \mathcal{A}_{[x \to u]}(F) = 1 \text{ for } \underline{\text{all}} / \underline{\text{some}} \; u \text{ in } U \\ 0 & \end{cases}$

L.6.7 For all F, G and H where x does not occur free in H

1) $\neg(\forall x F) \equiv \exists x \neg F$
2) $\neg(\exists x F) \equiv \forall x \neg F$
3) $(\forall x F) \wedge (\forall x G) \equiv \forall x (F \wedge G)$
4) $(\exists x F) \vee (\exists x G) \equiv \exists x (F \vee G)$
5) $\forall x \forall y F \equiv \forall y \forall x F$
6) $\exists x \exists y \equiv \exists y \exists x$
7) $(\forall x F) \wedge H \equiv \forall x (F \wedge H)$
8) $(\forall x F) \vee H \equiv \forall x (F \vee H)$
9) $(\exists x F) \wedge H \equiv \exists x (F \wedge H)$
10) $(\exists x F) \vee H \equiv \exists x (F \vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F

L.6.9 If y does not occur in G: (bound substitution)
$\forall x G \equiv \forall y G[x/y]$ and $\exists x G \equiv \exists y G[x/y]$

D.6.37 A formula in which no variable occurs both bound and free and in which all variables appearing after the quantifiers are distinct is in rectified form

D.6.38 A formula of the form $Q_1 x_1 Q_2 x_2 \dots Q_n x_n G$ where Q_i are quantifiers and G does not contain quantifiers is in prenex form
⇒ to achieve this, first rectified form, then L.6.7

T.6.10 For every formula there is an equiv. formula in prenex form

Prenex: ① rename bound variables ② remove all → ③ apply de Morgan ④ shift $\exists, \forall$ to front (L.6.7)

L.6.11 Universal instantiation: $\forall x F \models F[x/t]$ for any F, t

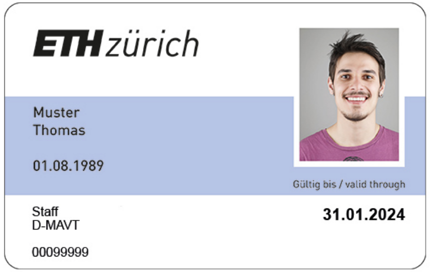
T.6.12 $\neg \exists x \forall y (P(y,x) \leftrightarrow \neg P(y,y))$ (is tautology)

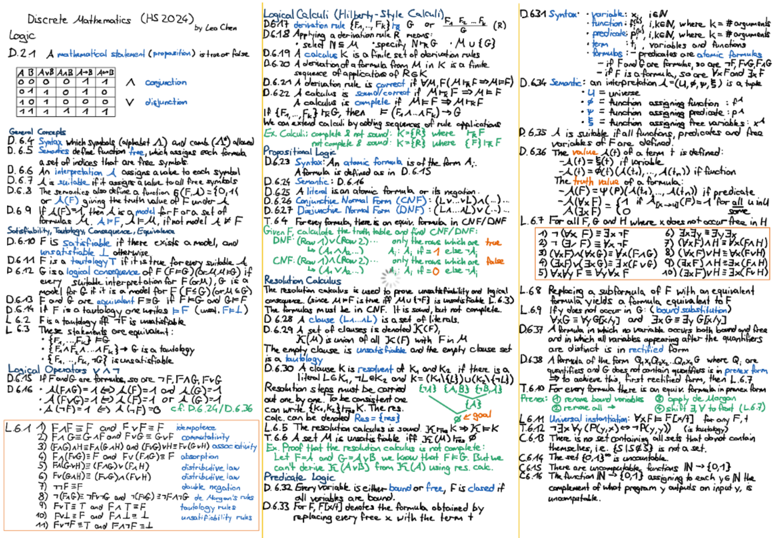
C.6.13 There is no set containing all sets that do not contain themselves, i.e. $\{S \mid S \notin S\}$ is not a set.

C.6.14 The set $\{0,1\}^\infty$ is uncountable.

C.6.15 There are uncomputable functions $\mathbb{N} \to \{0,1\}$

C.6.16 The function $\mathbb{N} \to \{0,1\}$ assigning to each $y \in \mathbb{N}$ the complement of what program y outputs on input y, is uncomputable.





Discrete Mathematics (HS 2024) by Leo Chen

Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	$A \vee B$	$A \wedge B$	$A \to B$	$A \leftrightarrow B$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

$\wedge$ conjunction
$\vee$ disjunction

General Concepts
D.6.4 Syntax which symbols (alphabet Λ) and comb. (Λ^*) allowed
D.6.5 Semantics define function free, which assigns each formula a set of indices that are free symbols.
D.6.6 An interpretation $\mathcal{A}$ assigns a value to each symbol
D.6.7 $\mathcal{A}$ is suitable if it assigns a value to all free symbols
D.6.8 The semantics also define a function $\sigma(F, \mathcal{A}) = \{0,1\}$ or $\mathcal{A}(F)$ giving the truth value of F under $\mathcal{A}$
D.6.9 If $\mathcal{A}(F)=1$, then $\mathcal{A}$ is a model for F or a set of formulas M. $\mathcal{A} \models F$, $\mathcal{A} \models M$, if not model $\mathcal{A} \not\models F$

Satisfiability, Tautology, Consequence, Equivalence
D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise
D.6.11 F is a tautology T if it is true for every suitable $\mathcal{A}$
D.6.12 G is a logical consequence of F ($F \models G$) (or M, $M \models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F \leq G$) (or $M \leq G$)
D.6.13 F and G are equivalent $F \equiv G$ if $F \models G$ and $G \models F$
D.6.14 If F is a tautology one writes $\models F$ (unsat. $F \models \perp$)
L.6.2 F is a tautology iff $\neg F$ is unsatisfiable
L.6.3 These statements are equivalent:
- $\{F_1, \dots, F_k\} \models G$
- $\{F_1 \wedge F_2 \wedge \dots \wedge F_k\} \to G$ is a tautology
- $\{F_1, \dots, F_k, \neg G\}$ is unsatisfiable

Logical Operators $\vee \wedge \neg$
D.6.15 If F and G are formulas, so are $\neg F$, $F \wedge G$, $F \vee G$
D.6.16
- $\mathcal{A}(F \wedge G) = 1 \Leftrightarrow \mathcal{A}(F)=1$ and $\mathcal{A}(G)=1$
- $\mathcal{A}(F \vee G) = 1 \Leftrightarrow \mathcal{A}(F)=1$ or $\mathcal{A}(G)=1$
- $\mathcal{A}(\neg F) = 1 \Leftrightarrow \mathcal{A}(\neg F) = 0$ cf. D.6.24/D.6.36

L.6.1
1) $F \wedge F \equiv F$ and $F \vee F \equiv F$ idempotence
2) $F \wedge G \equiv G \wedge F$ and $F \vee G \equiv G \vee F$ commutativity
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9) $F \vee T \equiv T$ and $F \wedge T \equiv F$ tautology rules
10) $F \vee \perp \equiv F$ and $F \wedge \perp \equiv \perp$ unsatifiability rules
11) $F \vee \neg F \equiv T$ and $F \wedge \neg F \equiv \perp$

Logical Calculi (Hilberty-Style Calculi)
D.6.17 derivation rule $\{F_1, \dots, F_k\} \vdash_R G$ or $\dfrac{F_1 \ F_2 \ \dots \ F_k}{G}$ (R)
D.6.18 Applying a derivation rule R means:
- select $N \subseteq M$ • specify $N \vdash_R G$ • $M \cup \{G\}$

D.6.19 A calculus K is a finite set of derivation rules
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D.6.21 A derivation rule is correct if $\forall M, F (M \vdash_R F \Rightarrow M \models F)$
D.6.22 A calculus is sound/correct if $M \vdash_K F \Rightarrow M \models F$
A calculus is complete if $M \models F \Rightarrow M \vdash_K F$
If $\{F_1, \dots, F_k\} \vdash_K G$, then $\models (F_1 \wedge \dots \wedge F_k) \to G$
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Ex. Calculi: complete & not sound: $K = \{R\}$ where $\vdash_R F$
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Propositional Logic
D.6.23 Syntax: An atomic formula is of the form A_i.
A formula is defined as in D.6.15
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D.6.26 Conjunctive Normal Form (CNF): $(L \vee \dots \vee L) \wedge (\dots) \dots$
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Resolution Calculus
The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M \models F$ is true iff $M \cup \{\neg F\}$ is unsatisfiable L.6.3)
The formulas must be in CNF. It is sound, but not complete.
D.6.28 A clause $(L \wedge \dots \wedge L)$ is a set of literals.
D.6.29 A set of clauses is denoted $\mathcal{K}(F)$,
$\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M
The empty clause is unsatisfiable and the empty clause set is a tautology
D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L \in K_1$, $\neg L \in K_2$ and $K = (K_1 \setminus \{L\}) \cup (K_2 \setminus \{\neg L\})$
Resolution steps must be carried out one by one. To be consistent one can write $\{K_1, K_2\} \vdash_{res} K$. The res. calc. can be denoted $Res = \{res\}$
$\{\neg A\}$ $\{A, B\}$ $\{\neg B, A\}$
$\{A\}$
$\emptyset$ ← goal
L.6.5 The resolution calculus is sound. $\mathcal{K} \vdash_{res} K \Rightarrow \mathcal{K} \models K$
T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M) \vdash_{res} \emptyset$
Ex. Proof that the resolution calculus is not complete:
Let $F = A$ and $G = A \vee B$ we know that $F \models G$. But we can't derive $\mathcal{K}(A \vee B)$ from $\mathcal{K}(A)$ using res. calc.

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- term: t_i, variables and functions
- formulas: – predicates are atomic formulas
  – if F and G are formulas, so are $\neg F$, $F \vee G$, $F \wedge G$
  – if F is a formula, so are $\forall x F$ and $\exists x F$

D.6.34 Semantic: an interpretation $\mathcal{A} = (U, \phi, \psi, \xi)$ is a tuple
- U = universe
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4) $(\exists x F) \vee (\exists x G) \equiv \exists x (F \vee G)$
5) $\forall x \forall y F \equiv \forall y \forall x F$
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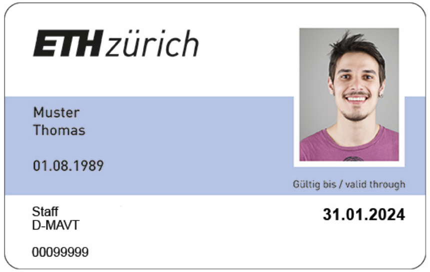

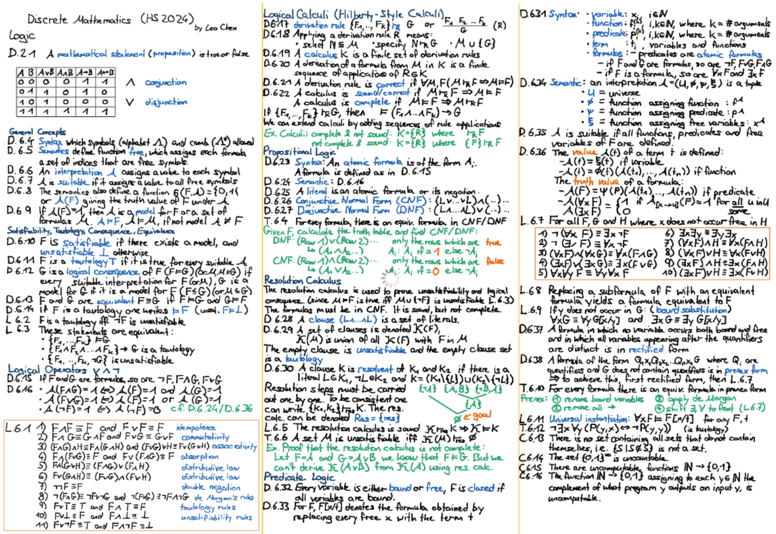

Discrete Mathematics (HS 2024)

by Leo Chen

Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	A∨B	A∧B	A→B	A↔B
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

$\wedge$ conjunction

$\vee$ disjunction

General Concepts

D.6.4 Syntax which symbols (alphabet Λ) and comb. (Λ^*) allowed

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D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise

D.6.11 F is a tautology $\top$ if it is true for every suitable $\mathcal{A}$

D.6.12 G is a logical consequence of F ($F \models G$) (or M, $M \models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F \leq G$) (or $M \leq G$)

D.6.13 F and G are equivalent $F \equiv G$ if $F \models G$ and $G \models F$

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- $\{F_1,\dots,F_k\} \models G$
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Logical Operators $\vee \wedge \neg$

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D.6.16
- $\mathcal{A}(F\wedge G)=1 \iff \mathcal{A}(F)=1$ and $\mathcal{A}(G)=1$
- $\mathcal{A}(F\vee G)=1 \iff \mathcal{A}(F)=1$ or $\mathcal{A}(G)=1$
- $\mathcal{A}(\neg F)=1 \iff \mathcal{A}(\neg F)=0$ c.f. D.6.24/D.6.36

L.6.1
1) $F\wedge F \equiv F$ and $F \vee F \equiv F$ idempotence
2) $F\wedge G \equiv G \wedge F$ and $F\vee G \equiv G \vee F$ commutativity
3) $(F\wedge G)\wedge H \equiv F\wedge(G\wedge H)$ and $(F\vee G)\vee H \equiv F\vee(G\vee H)$ associativity
4) $F\wedge(F\vee G)\equiv F$ and $F\vee(F\wedge G)\equiv F$ absorption
5) $F\wedge(G\vee H)\equiv(F\wedge G)\vee(F\wedge H)$ distributive law
6) $F\vee(G\wedge H)\equiv(F\vee G)\wedge(F\vee H)$ distributive law
7) $\neg\neg F \equiv F$ double negation
8) $\neg(F\wedge G)\equiv \neg F\vee\neg G$ and $\neg(F\vee G)\equiv\neg F\wedge\neg G$ de Morgan's rules
9) $F\vee\top\equiv\top$ and $F\wedge\top\equiv F$ tautology rules
10) $F\vee\perp\equiv F$ and $F\wedge\perp\equiv\perp$ unsatisfiability rules
11) $F\vee\neg F\equiv\top$ and $F\wedge\neg F\equiv\perp$

Logical Calculi (Hilberty-Style Calculi)

D.6.17 derivation rule $\{F_1,\dots,F_k\} \vdash_R G$ or $\dfrac{F_1 \; F_2 \; \dots \; F_k}{G}$ (R)

D.6.18 Applying a derivation rule R means:
- select $N \subseteq M$
- specify $N \vdash_R G$
- $M \cup \{G\}$

D.6.19 A calculus K is a finite set of derivation rules

D.6.20 A derivation of a formula from M in K is a finite sequence of applications of $R \in K$

D.6.21 A derivation rule is correct if $\forall M, F (M \vdash_R F \Rightarrow M \models F)$

D.6.22 A calculus is sound/correct if $M \vdash_K F \Rightarrow M \models F$
A calculus is complete if $M \models F \Rightarrow M \vdash_K F$

If $\{F_1,\dots,F_k\} \vdash_K G$, then $\models (F_1\wedge\dots\wedge F_k)\rightarrow G$

We can extend calculi by adding sequences of rule applications

Ex. Calculi: complete & not sound: $K=\{R\}$ where $\vdash_R F$
not complete & sound: $K=\{R\}$ where $\{F\} \vdash_R F$

Propositional Logic

D.6.23 Syntax: An atomic formula is of the form A_i.
A formula is defined as in D.6.15

D.6.24 Semantic: D.6.16

D.6.25 A literal is an atomic formula or its negation.

D.6.26 Conjunctive Normal Form (CNF): $(L\vee\dots\vee L)\wedge(\dots)\dots$

D.6.27 Disjunctive Normal Form (DNF): $(L\wedge\dots\wedge L)\vee(\dots)\dots$

T.6.4 For every formula, there is an equiv. formula in CNF/DNF

Given F, calculate the truth table and find CNF/DNF:
DNF: (Row 1)∨(Row 2)… only the rows which are true
↳ $(A_1 \wedge A_2 \dots)$ A_i: A_i if $=1$ else $\neg A_i$
CNF: (Row 1)∧(Row 2)… only the rows which are false
↳ $(A_1 \vee A_2 \dots)$ A_i: A_i if $=0$ else $\neg A_i$

Resolution Calculus

The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M \models F$ is true iff $M \cup \{\neg F\}$ is unsatisfiable L.6.3) The formulas must be in CNF. It is sound, but not complete.

D.6.28 A clause $(L\wedge\dots\wedge L)$ is a set of literals.

D.6.29 A set of clauses is denoted $\mathcal{K}(F)$,
$\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M

The empty clause is unsatisfiable and the empty clause set is a tautology

D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L \in K_1$, $\neg L \in K_2$ and $K = (K_1 \setminus \{L\}) \cup (K_2 \setminus \{\neg L\})$

Resolution steps must be carried out one by one. To be consistent one can write $\{K_1,K_2\} \vdash_{res} K$. The res. calc. can be denoted $Res = \{res\}$

$\{\neg A\}$ $\{A,B\}$ $\{\neg B, A\}$ → $\{A\}$ → $\emptyset$ ← goal

L.6.5 The resolution calculus is sound. $\mathcal{K} \vdash_{res} K \Rightarrow \mathcal{K} \models K$

T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M) \vdash_{res} \emptyset$

Ex. Proof that the resolution calculus is not complete:
Let $F=A$ and $G=A\vee B$ we know that $F\models G$. But we can't derive $\mathcal{K}(A\vee B)$ from $\mathcal{K}(A)$ using res. calc.

Predicate Logic

D.6.32 Every variable is either bound or free, F is closed if all variables are bound.

D.6.33 For F, $F[x/t]$ denotes the formula obtained by replacing every free x with the term t

D.6.31 Syntax:
- variable: x_i, $i\in\mathbb{N}$
- function: $f_i^{(k)}$, $i,k\in\mathbb{N}$, where k = #arguments
- predicate: $P_i^{(k)}$, $i,k\in\mathbb{N}$, where k = #arguments
- term: t_i, variables and functions
- formulas:
  - predicates are atomic formulas
  - if F and G are formulas, so are $\neg F$, $F\vee G$, $F\wedge G$
  - if F is a formula, so are $\forall x F$ and $\exists x F$

D.6.34 Semantic: an interpretation $\mathcal{A}=(U,\phi,\psi,\xi)$ is a tuple
- U = universe
- ϕ = function assigning function : $f^{\mathcal{A}}$
- ψ = function assigning predicate : $P^{\mathcal{A}}$
- ξ = function assigning free variables : $x^{\mathcal{A}}$

D.6.35 $\mathcal{A}$ is suitable if all functions, predicates and free variables of F are defined.

D.6.36 The value $\mathcal{A}(t)$ of a term t is defined:
- $\mathcal{A}(t)=\xi(t)$ if variable
- $\mathcal{A}(t)=\phi(f)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if function

The truth value of a formula:
- $\mathcal{A}(F)=\psi(P)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if predicate
- $\mathcal{A}(\forall x F) = \begin{cases}1 & \text{if } \mathcal{A}_{[x\to u]}(F)=1 \text{ for all } u \text{ in } U\\ 0\end{cases}$, $\mathcal{A}(\exists x F)$ … some

L.6.7 For all F, G and H where x does not occur free in H
1) $\neg(\forall x F) \equiv \exists x \neg F$
2) $\neg(\exists x F) \equiv \forall x \neg F$
3) $(\forall x F)\wedge(\forall x G) \equiv \forall x (F\wedge G)$
4) $(\exists x F)\vee(\exists x G) \equiv \exists x (F\vee G)$
5) $\forall x \forall y F \equiv \forall y \forall x F$
6) $\exists x \exists y \equiv \exists y \exists x$
7) $(\forall x F)\wedge H \equiv \forall x (F\wedge H)$
8) $(\forall x F)\vee H \equiv \forall x (F\vee H)$
9) $(\exists x F)\wedge H \equiv \exists x (F\wedge H)$
10) $(\exists x F)\vee H \equiv \exists x (F\vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F

L.6.9 If y does not occur in G (bound substitution) $\forall x G \equiv \forall y G[x/y]$ and $\exists x G \equiv \exists y G[x/y]$

D.6.37 A formula in which no variable occurs both bound and free and in which all variables appearing after the quantifiers are distinct is in rectified form

D.6.38 A formula of the form $Q_1x_1 Q_2x_2 \dots Q_nx_n G$ where Q_i are quantifiers and G does not contain quantifiers is in prenex form
⇒ to achieve this, first rectified form, then L.6.7

T.6.10 For every formula there is an equiv. formula in prenex form

Prenex: ① rename bound variables ② remove all → ③ apply de Morgan ④ shift ∃∀ to front (L.6.7)

L.6.11 Universal instantiation: $\forall x F \models F[x/t]$ for any F, t

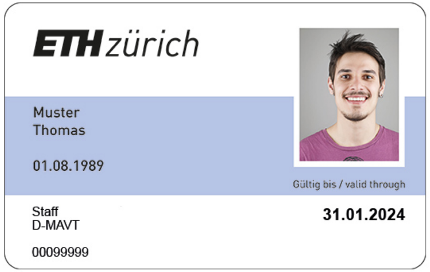
T.6.12 $\neg\exists x \forall y (P(y,x) \Leftrightarrow \neg P(y,y))$ (is tautology)

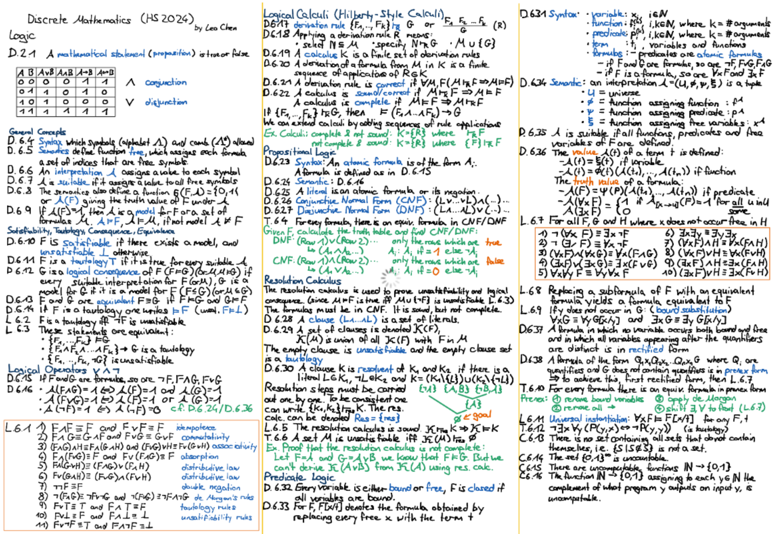
C.6.13 There is no set containing all sets that do not contain themselves, i.e. $\{S \mid S \notin S\}$ is not a set.

C.6.14 The set $\{0,1\}^\infty$ is uncountable.

C.6.15 There are uncomputable functions $\mathbb{N}\rightarrow\{0,1\}$

C.6.16 The function $\mathbb{N}\rightarrow\{0,1\}$ assigning to each $y\in\mathbb{N}$ the complement of what program y outputs on input y, is uncomputable.





Discrete Mathematics (HS 2024) by Leo Chen

Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	A∨B	A∧B	A→B	A↔B
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

$\wedge$ conjunction
$\vee$ disjunction

General Concepts
D.6.4 Syntax which symbols (alphabet Λ) and comb. (Λ^*) allowed
D.6.5 Semantics define function free, which assigns each formula a set of indices that are free symbols.
D.6.6 An interpretation $\mathcal{A}$ assigns a value to each symbol
D.6.7 $\mathcal{A}$ is suitable if it assigns a value to all free symbols
D.6.8 The semantics also define a function $\sigma(F,\mathcal{A}) = \{0,1\}$ or $\mathcal{A}(F)$ giving the truth value of F under $\mathcal{A}$
D.6.9 If $\mathcal{A}(F)=1$, then $\mathcal{A}$ is a model for F or a set of formulas M. $\mathcal{A} \models F$, $\mathcal{A} \models M$, if not model $\mathcal{A} \not\models F$

Satisfiability, Tautology, Consequence, Equivalence
D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise
D.6.11 F is a tautology T if it is true for every suitable $\mathcal{A}$
D.6.12 G is a logical consequence of F ($F \models G$) (or M, $M \models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F \leq G$) (or $M \leq G$)
D.6.13 F and G are equivalent $F \equiv G$ if $F \models G$ and $G \models F$
D.6.14 If F is a tautology one writes $\models F$ (unsat. $F \models \perp$)
L.6.2 F is a tautology iff $\neg F$ is unsatisfiable
L.6.3 These statements are equivalent:
- $\{F_1, \ldots, F_k\} \models G$
- $\{F_1 \wedge F_2 \wedge \ldots \wedge F_k\} \to G$ is a tautology
- $\{F_1, \ldots, F_k, \neg G\}$ is unsatisfiable

Logical Operators $\vee \wedge \neg$
D.6.15 If F and G are formulas, so are $\neg F$, $F \wedge G$, $F \vee G$
D.6.16
- $\mathcal{A}(F \wedge G) = 1 \iff \mathcal{A}(F) = 1$ and $\mathcal{A}(G) = 1$
- $\mathcal{A}(F \vee G) = 1 \iff \mathcal{A}(F) = 1$ or $\mathcal{A}(G) = 1$
- $\mathcal{A}(\neg F) = 1 \iff \mathcal{A}(\neg F) = 0$ cf. D.6.24/D.6.36

L.6.1
1) $F \wedge F \equiv F$ and $F \vee F \equiv F$ idempotence
2) $F \wedge G \equiv G \wedge F$ and $F \vee G \equiv G \vee F$ commutativity
3) $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$ and $(F \vee G) \vee H \equiv F \vee (G \vee H)$ associativity
4) $F \wedge (F \vee G) \equiv F$ and $F \vee (F \wedge G) \equiv F$ absorption
5) $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ distributive law
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9) $F \vee T \equiv T$ and $F \wedge T \equiv F$ tautology rules
10) $F \vee \perp \equiv F$ and $F \wedge \perp \equiv \perp$ unsatifiability rules
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Logical Calculi (Hilberty-Style Calculi)
D.6.17 derivation rule $\{F_1, \ldots, F_k\} \vdash_R G$ or $\frac{F_1 \ F_2 \ \ldots \ F_k}{G}$ (R)
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Resolution Calculus
The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M \models F$ is true iff $M \cup \{\neg F\}$ is unsatisfiable L.6.3)
The formulas must be in CNF. It is sound, but not complete.
D.6.28 A clause $(L \wedge \ldots \wedge L)$ is a set of literals.
D.6.29 A set of clauses is denoted $\mathcal{K}(F)$,
$\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M
The empty clause is unsatisfiable and the empty clause set is a tautology
D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L \in K_1$, $\neg L \in K_2$ and $K = (K_1 \setminus \{L\}) \cup (K_2 \setminus \{\neg L\})$
Resolution steps must be carried out one by one. To be consistent one can write $\{K_1, K_2\} \vdash_{res} K$. The res. calc. can be denoted $Res = \{res\}$

$\{\neg A\}$ $\{A, B\}$ $\{\neg B, A\}$
$\{A\}$
$\emptyset$ ← goal

L.6.5 The resolution calculus is sound. $\mathcal{K} \vdash_{res} K \Rightarrow \mathcal{K} \models K$
T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M) \vdash_{res} \emptyset$
Ex. Proof that the resolution calculus is not complete:
Let $F = A$ and $G = A \vee B$ we know that $F \models G$. But we can't derive $\mathcal{K}(A \vee B)$ from $\mathcal{K}(A)$ using res. calc.

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- formulas:
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- U = universe
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- $\mathcal{A}(t) = \xi(t)$ if variable
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The truth value of a formula:
- $\mathcal{A}(F) = \psi(P)(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n))$ if predicate
- $\mathcal{A}(\forall x F)$, $\mathcal{A}(\exists x F)$ $= \begin{cases} 1 & \text{if } \mathcal{A}_{[x \to u]}(F) = 1 \text{ for all / some } u \text{ in } U \\ 0 & \end{cases}$

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1) $\neg(\forall x F) \equiv \exists x \neg F$
2) $\neg(\exists x F) \equiv \forall x \neg F$
3) $(\forall x F) \wedge (\forall x G) \equiv \forall x (F \wedge G)$
4) $(\exists x F) \vee (\exists x G) \equiv \exists x (F \vee G)$
5) $\forall x \forall y F \equiv \forall y \forall x F$
6) $\exists x \exists y \equiv \exists y \exists x$
7) $(\forall x F) \wedge H \equiv \forall x (F \wedge H)$
8) $(\forall x F) \vee H \equiv \forall x (F \vee H)$
9) $(\exists x F) \wedge H \equiv \exists x (F \wedge H)$
10) $(\exists x F) \vee H \equiv \exists x (F \vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F
L.6.9 If y does not occur in G: (bound substitution)
$\forall x G \equiv \forall y G[x/y]$ and $\exists x G \equiv \exists y G[x/y]$
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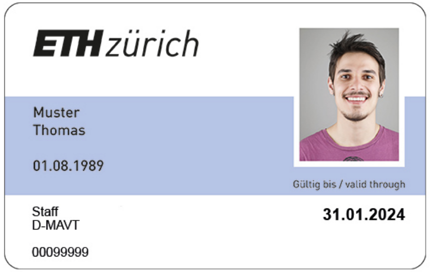

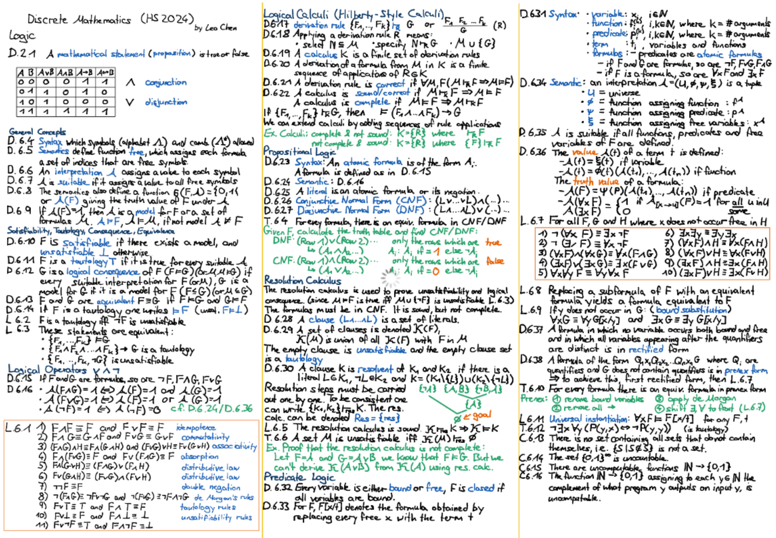
Discrete Mathematics (HS 2024) by Leo Chen

Logic

A	B	A∨B	A∧B	A→B	A↔B
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
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General Concepts

Satisfiability, Tautology, Consequence, Equivalence

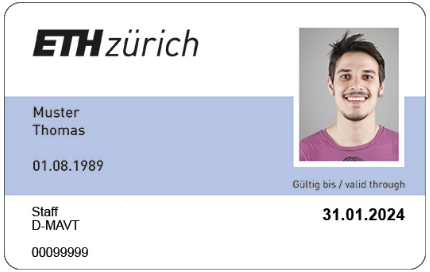
Logical Operators

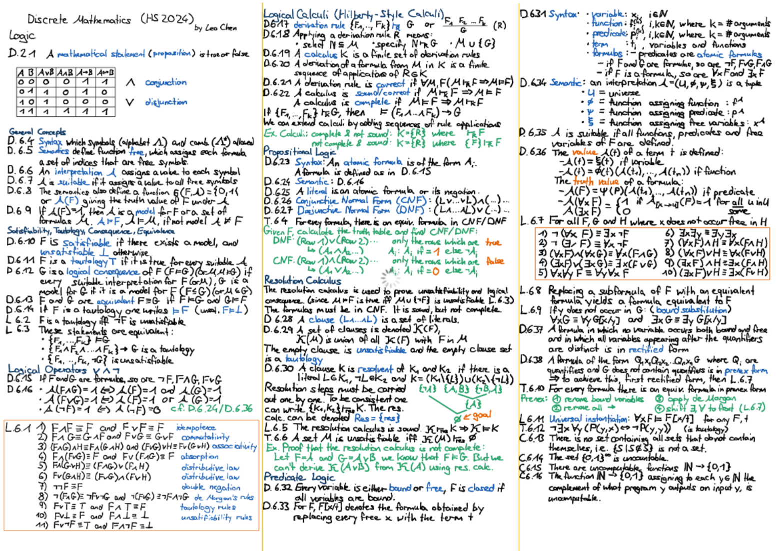
Logical Calculi (Hilbert-Style Calculi)

Propositional Logic

Resolution Calculus

Predicate Logic





Discrete Mathematics (HS 2024)

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Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	A∨B	A∧B	A→B	A↔B
0	0	0	0	1	1
0	1	1	0	1	0
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∧ conjunction

∨ disjunction

General Concepts

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D.6.12 G is a logical consequence of F ($F \models G$) (or M, $M \models G$) if every suitable interpretation for F (or M), G is a model for G if it is a model for F ($F \leq G$) (or $M \leq G$)

D.6.13 F and G are equivalent $F \equiv G$ if $F \models G$ and $G \models F$

D.6.14 If F is a tautology one writes $\models F$ (unsat. $F \models \bot$)

L.6.2 F is a tautology iff $\neg F$ is unsatisfiable

L.6.3 These statements are equivalent:
- $\{F_1, \dots, F_k\} \models G$
- $\{F_1 \wedge F_2 \wedge \dots \wedge F_k\} \to G$ is a tautology
- $\{F_1, \dots, F_k, \neg G\}$ is unsatisfiable

Logical Operators ∨ ∧ ¬

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- $\mathcal{A}(F \wedge G) = 1 \iff \mathcal{A}(F)=1$ and $\mathcal{A}(G)=1$
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- $\mathcal{A}(\neg F) = 1 \iff \mathcal{A}(\neg F) = 0$ cf. D.6.24/D.6.36

L.6.1
1) $F \wedge F \equiv F$ and $F \vee F \equiv F$ idempotence
2) $F \wedge G \equiv G \wedge F$ and $F \vee G \equiv G \vee F$ commutativity
3) $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$ and $(F \vee G) \vee H \equiv F \vee (G \vee H)$ associativity
4) $F \wedge (F \vee G) \equiv F$ and $F \vee (F \wedge G) \equiv F$ absorption
5) $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$ distributive law
6) $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$ distributive law
7) $\neg\neg F \equiv F$ double negation
8) $\neg(F \wedge G) \equiv \neg F \vee \neg G$ and $\neg(F \vee G) \equiv \neg F \wedge \neg G$ de Morgan's rules
9) $F \vee T \equiv T$ and $F \wedge T \equiv F$ tautology rules
10) $F \vee \bot \equiv F$ and $F \wedge \bot \equiv \bot$ unsatifiability rules
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Logical Calculi (Hilberty-Style Calculi)

D.6.17 derivation rule $\{F_1, \dots, F_k\} \vdash_R G$ or $\frac{F_1 \quad F_2 \quad \dots \quad F_k}{G}$ (R)

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D.6.21 A derivation rule is correct if $\forall M, F (M \vdash_R F \Rightarrow M \models F)$

D.6.22 A calculus is sound/correct if $M \vdash_K F \Rightarrow M \models F$
A calculus is complete if $M \models F \Rightarrow M \vdash_K F$

If $\{F_1, \dots, F_k\} \vdash_K G$, then $\models (F_1 \wedge \dots \wedge F_k) \to G$

We can extend calculi by adding sequences of rule applications

Ex. Calculi: complete & not sound: $K = \{R\}$ where $\vdash_R F$
not complete & sound: $K = \{R\}$ where $\{F\} \vdash_R F$

Propositional Logic

D.6.23 Syntax: An atomic formula is of the form A_i.
A formula is defined as in D.6.15

D.6.24 Semantic: D.6.16

D.6.25 A literal is an atomic formula or its negation.

D.6.26 Conjunctive Normal Form (CNF): $(L \vee \dots \vee L) \wedge (\dots) \dots$

D.6.27 Disjunctive Normal Form (DNF): $(L \wedge \dots \wedge L) \vee (\dots) \dots$

T.6.4 For every formula, there is an equiv. formula in CNF/DNF

Given F, calculate the truth table and find CNF/DNF:
DNF: (Row 1) ∨ (Row 2) … only the rows which are true
↳ $(A_1 \wedge A_2 \dots)$ A_i: A_i if $= 1$ else $\neg A_i$
CNF: (Row 1) ∧ (Row 2) … only the rows which are false
↳ $(A_1 \vee A_2 \dots)$ A_i: A_i if $= 0$ else $\neg A_i$

Resolution Calculus

The resolution calculus is used to prove unsatisfiability and logical consequence. (since $M \models F$ is true iff $M \cup \{\neg F\}$ is unsatisfiable L.6.3) The formulas must be in CNF. It is sound, but not complete.

D.6.28 A clause $(L \wedge \dots \wedge L)$ is a set of literals.

D.6.29 A set of clauses is denoted $\mathcal{K}(F)$,
$\mathcal{K}(M)$ is union of all $\mathcal{K}(F)$ with F in M

The empty clause is unsatisfiable and the empty clause set is a tautology

D.6.30 A clause K is resolvent of K_1 and K_2 if there is a literal $L \in K_1$, $\neg L \in K_2$ and $K = (K_1 \setminus \{L\}) \cup (K_2 \setminus \{\neg L\})$

Resolution steps must be carried out one by one. To be consistent one can write $\{K_1, K_2\} \vdash_{res} K$. The res. calc. can be denoted $Res = \{res\}$

$\{\neg A\}$ $\{A, B\}$ $\{\neg B, A\}$
$\{A\}$
$\emptyset$ ← goal

L.6.5 The resolution calculus is sound. $\mathcal{K} \vdash_{res} K \Rightarrow \mathcal{K} \models K$

T.6.6 A set M is unsatisfiable iff $\mathcal{K}(M) \vdash_{res} \emptyset$

Ex. Proof that the resolution calculus is not complete:
Let $F = A$ and $G = A \vee B$ we know that $F \models G$. But we can't derive $\mathcal{K}(A \vee B)$ from $\mathcal{K}(A)$ using res. calc.

Predicate Logic

D.6.32 Every variable is either bound or free, F is closed if all variables are bound.

D.6.33 For F, F[x/t] denotes the formula obtained by replacing every free x with the term t

D.6.31 Syntax:
- variable: x_i, $i \in \mathbb{N}$
- function: $f_i^{(k)}$, $i, k \in \mathbb{N}$, where k = # arguments
- predicate: $P_i^{(k)}$, $i, k \in \mathbb{N}$, where k = # arguments
- term: t_i: variables and functions
- formulas:
  – predicates are atomic formulas
  – if F and G are formulas, so are $\neg F$, $F \vee G$, $F \wedge G$
  – if F is a formula, so are $\forall x F$ and $\exists x F$

D.6.34 Semantic: an interpretation $\mathcal{A} = (U, \phi, \psi, \xi)$ is a tuple
- U = universe
- φ = function assigning function: $f^{\mathcal{A}}$
- ψ = function assigning predicate: $P^{\mathcal{A}}$
- ξ = function assigning free variables: $x^{\mathcal{A}}$

D.6.35 $\mathcal{A}$ is suitable if all functions, predicates and free variables of F are defined.

D.6.36 The value $\mathcal{A}(t)$ of a term t is defined:
– $\mathcal{A}(t) = \xi(t)$ if variable
– $\mathcal{A}(t) = \phi(t)(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))$ if function
The truth value of a formula:
– $\mathcal{A}(F) = \psi(P)(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))$ if predicate
– $\mathcal{A}(\forall x F) = 1$ if $\mathcal{A}_{[x \to u]}(F) = 1$ for all u in U, 0 otherwise
– $\mathcal{A}(\exists x F) = 1$ for some u, 0 otherwise

L.6.7 For all F, G and H where x does not occur free in H
1) $\neg(\forall x F) \equiv \exists x \neg F$
2) $\neg(\exists x F) \equiv \forall x \neg F$
3) $(\forall x F) \wedge (\forall x G) \equiv \forall x (F \wedge G)$
4) $(\exists x F) \vee (\exists x G) \equiv \exists x (F \vee G)$
5) $\forall x \forall y F \equiv \forall y \forall x F$
6) $\exists x \exists y \equiv \exists y \exists x$
7) $(\forall x F) \wedge H \equiv \forall x (F \wedge H)$
8) $(\forall x F) \vee H \equiv \forall x (F \vee H)$
9) $(\exists x F) \wedge H \equiv \exists x (F \wedge H)$
10) $(\exists x F) \vee H \equiv \exists x (F \vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F

L.6.9 If y does not occur in G (bound substitution) $\forall x G \equiv \forall y G[x/y]$ and $\exists x G \equiv \exists y G[x/y]$

D.6.37 A formula in which no variable occurs both bound and free and in which all variables appearing after the quantifiers are distinct is in rectified form

D.6.38 A formula of the form $Q_1 x_1 Q_2 x_2 \dots Q_n x_n G$ where Q are quantifiers and G does not contain quantifiers is in prenex form
⇒ to achieve this, first rectified form, then L.6.7

T.6.10 For every formula there is an equiv. formula in prenex form

Prenex: ① rename bound variables ③ apply de Morgan
② rename all → ④ shift ∃∀ to front (L.6.7)

L.6.11 Universal instantiation: $\forall x F \models F[x/t]$ for any F, t

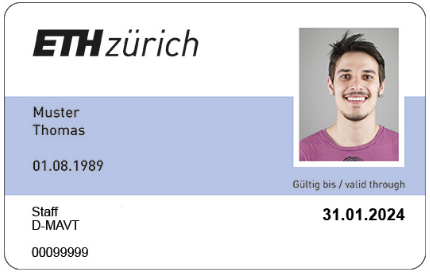
T.6.12 $\neg \exists x \forall y (P(y,x) \leftrightarrow \neg P(y,y))$ (is tautology)

C.6.13 There is no set containing all sets that do not contain themselves, i.e. $\{S \mid S \notin S\}$ is not a set.

C.6.14 The set $\{0,1\}^\infty$ is uncountable.

C.6.15 There are uncomputable functions $\mathbb{N} \to \{0,1\}$

C.6.16 The function $\mathbb{N} \to \{0,1\}$ assigning to each $y \in \mathbb{N}$ the complement of what program y outputs on input y, is uncomputable.



Discrete Mathematics (HS 2024)

by Leo Chen

Logic

D.2.1 A mathematical statement (proposition) is true or false

A	B	$A\vee B$	$A\wedge B$	$A\to B$	$A\leftrightarrow B$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

$\wedge$ conjunction

$\vee$ disjunction

General Concepts

D.6.4 Syntax which symbols (alphabet Λ) and comb. (Λ^*) allowed

D.6.5 Semantics define function free, which assigns each formula a set of indices that are free symbols.

D.6.6 An interpretation $\mathcal{A}$ assigns a value to each symbol

D.6.7 $\mathcal{A}$ is suitable if it assigns a value to all free symbols

D.6.8 The semantics also define a function $\sigma(F,\mathcal{A})=\{0,1\}$ or $\mathcal{A}(F)$ giving the truth value of F under $\mathcal{A}$

D.6.9 If $\mathcal{A}(F)=1$, then $\mathcal{A}$ is a model for F or a set of formulas $\mathcal{M}$. $\mathcal{A}\models F$, $\mathcal{A}\models\mathcal{M}$, if not model $\mathcal{A}\not\models F$

Satisfiability, Tautology, Consequence, Equivalence

D.6.10 F is satisfiable if there exists a model, and unsatisfiable $\perp$ otherwise

D.6.11 F is a tautology $\top$ if it is true for every suitable $\mathcal{A}$

D.6.12 G is a logical consequence of F ($F\models G$) (or $\mathcal{M}$, $\mathcal{M}\models G$) if every suitable interpretation for F (or $\mathcal{M}$), G is a model for G if it is a model for F ($F\leq G$) (or $\mathcal{M}\leq G$)

D.6.13 F and G are equivalent $F\equiv G$ if $F\models G$ and $G\models F$

D.6.14 If F is a tautology one writes $\models F$ (unsat. $F\models\perp$)

L.6.2 F is a tautology iff $\neg F$ is unsatisfiable

L.6.3 These statements are equivalent:
- $\{F_1,\dots,F_k\}\models G$
- $\{F_1\wedge F_2\wedge\dots\wedge F_k\}\to G$ is a tautology
- $\{F_1,\dots,F_k,\neg G\}$ is unsatisfiable

Logical Operators $\vee\wedge\neg$

D.6.15 If F and G are formulas, so are $\neg F, F\wedge G, F\vee G$

D.6.16
- $\mathcal{A}(F\wedge G)=1 \Leftrightarrow \mathcal{A}(F)=1$ and $\mathcal{A}(G)=1$
- $\mathcal{A}(F\vee G)=1 \Leftrightarrow \mathcal{A}(F)=1$ or $\mathcal{A}(G)=1$
- $\mathcal{A}(\neg F)=1 \Leftrightarrow \mathcal{A}(\neg F)=0$ cf. D.6.24/D.6.36

L.6.1
1) $F\wedge F\equiv F$ and $F\vee F\equiv F$ idempotence
2) $F\wedge G\equiv G\wedge F$ and $F\vee G\equiv G\vee F$ commutativity
3) $(F\wedge G)\wedge H\equiv F\wedge(G\wedge H)$ and $(F\vee G)\vee H\equiv F\vee(G\vee H)$ associativity
4) $F\wedge(F\vee G)\equiv F$ and $F\vee(F\wedge G)\equiv F$ absorption
5) $F\wedge(G\vee H)\equiv(F\wedge G)\vee(F\wedge H)$ distributive law
6) $F\vee(G\wedge H)\equiv(F\vee G)\wedge(F\vee H)$ distributive law
7) $\neg\neg F\equiv F$ double negation
8) $\neg(F\wedge G)\equiv\neg F\vee\neg G$ and $\neg(F\vee G)\equiv\neg F\wedge\neg G$ de Morgan's rules
9) $F\vee\top\equiv\top$ and $F\wedge\top\equiv F$ tautology rules
10) $F\vee\perp\equiv F$ and $F\wedge\perp\equiv\perp$ unsatifiability rules
11) $F\vee\neg F\equiv\top$ and $F\wedge\neg F\equiv\perp$

Logical Calculi (Hilberty-Style Calculi)

D.6.17 derivation rule $\{F_1,\dots,F_k\}\vdash_R G$ or $\dfrac{F_1\ F_2\ \dots\ F_k}{G}$ (R)

D.6.18 Applying a derivation rule R means:
- select $N\subseteq\mathcal{M}$ • specify $N\vdash_R G$ • $\mathcal{M}\cup\{G\}$

D.6.19 A calculus K is a finite set of derivation rules

D.6.20 A derivation of a formula from $\mathcal{M}$ in K is a finite sequence of applications of $R\in K$

D.6.21 A derivation rule is correct if $\forall\mathcal{M},F(\mathcal{M}\vdash_R F\Rightarrow\mathcal{M}\models F)$

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A calculus is complete if $\mathcal{M}\models F\Rightarrow\mathcal{M}\vdash_K F$

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Resolution steps must be carried out one by one. To be consistent one can write $\{K_1,K_2\}\vdash_{res}K$. The res. calc. can be denoted $Res=\{res\}$

$\{\neg A\}$ $\{A,B\}$ $\{\neg B,A\}$ → $\{A\}$ → $\emptyset$ ← goal

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$\Rightarrow$ to achieve this, first rectified form, then L.6.7

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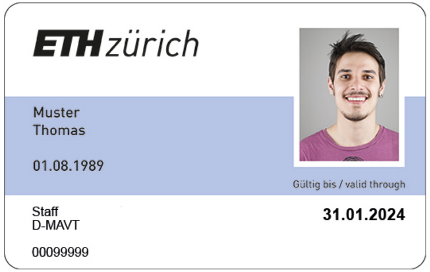

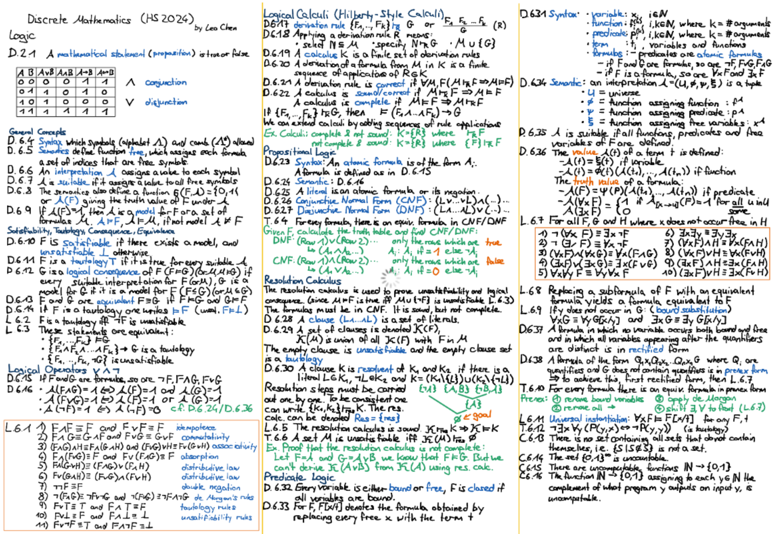

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$\wedge$ conjunction
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T.6.6 A set $\mathcal{M}$ is unsatisfiable iff $\mathcal{K}(\mathcal{M})\vdash_{res}\emptyset$
Ex. Proof that the resolution calculus is not complete:
Let $F=A$ and $G=A\vee B$ we know that $F\models G$. But we can't derive $\mathcal{K}(A\vee B)$ from $\mathcal{K}(A)$ using res. calc.

Predicate Logic
D.6.32 Every variable is either bound or free, F is closed if all variables are bound.
D.6.33 For F, $F[x/t]$ denotes the formula obtained by replacing every free x with the term t

D.6.31 Syntax:
- variable: x_i, $i\in\mathbb{N}$
- function: $f_i^{(k)}$, $i,k\in\mathbb{N}$, where $k=\#$arguments
- predicate: $P_i^{(k)}$, $i,k\in\mathbb{N}$, where $k=\#$arguments
- term: t_i, variables and functions
- formulas: – predicates are atomic formulas
  – if F and G are formulas, so are $\neg F$, $F\vee G$, $F\wedge G$
  – if F is a formula, so are $\forall x F$ and $\exists x F$

D.6.34 Semantic: an interpretation $\mathcal{A}=(U,\phi,\psi,\xi)$ is a tuple
- U = universe
- ϕ = function assigning function: $f^{\mathcal{A}}$
- ψ = function assigning predicate: $P^{\mathcal{A}}$
- ξ = function assigning free variables: $x^{\mathcal{A}}$

D.6.35 $\mathcal{A}$ is suitable if all functions, predicates and free variables of F are defined.
D.6.36 The value $\mathcal{A}(t)$ of a term t is defined:
– $\mathcal{A}(t)=\xi(t)$ if variable
– $\mathcal{A}(t)=\phi(t)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if function
The truth value of a formula:
– $\mathcal{A}(F)=\psi(P)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$ if predicate
– $\mathcal{A}(\forall x F)$, $\mathcal{A}(\exists x F)$ $=\begin{cases}1 & \text{if } \mathcal{A}_{[x\to u]}(F)=1 \text{ for all / some } u \text{ in } U\\ 0\end{cases}$
L.6.7 For all F, G and H where x does not occur free in H

1) $\neg(\forall x F)\equiv\exists x\neg F$
2) $\neg(\exists x F)\equiv\forall x\neg F$
3) $(\forall x F)\wedge(\forall x G)\equiv\forall x(F\wedge G)$
4) $(\exists x F)\vee(\exists x G)\equiv\exists x(F\vee G)$
5) $\forall x\forall y F\equiv\forall y\forall x F$
6) $\exists x\exists y\equiv\exists y\exists x$
7) $(\forall x F)\wedge H\equiv\forall x(F\wedge H)$
8) $(\forall x F)\vee H\equiv\forall x(F\vee H)$
9) $(\exists x F)\wedge H\equiv\exists x(F\wedge H)$
10) $(\exists x F)\vee H\equiv\exists x(F\vee H)$

L.6.8 Replacing a subformula of F with an equivalent formula yields a formula equivalent to F
L.6.9 If y does not occur in G: (bound substitution)
$\forall x G\equiv\forall y G[x/y]$ and $\exists x G\equiv\exists y G[x/y]$
D.6.37 A formula in which no variable occurs both bound and free and in which all variables appearing after the quantifiers are distinct is in rectified form
D.6.38 A formula of the form $Q_1x_1Q_2x_2\dots Q_nx_nG$ where Q_i are quantifiers and G does not contain quantifiers is in prenex form
⇒ to achieve this, first rectified form, then L.6.7
T.6.10 For every formula there is an equiv. formula in prenex form
Prenex: ① rename bound variables ② remove all → ③ apply de Morgan ④ shift ∃, ∀ to front (L.6.7)
L.6.11 Universal instantiation: $\forall x F\models F[x/t]$ for any F, t
T.6.12 $\neg\exists x\forall y(P(y,x)\leftrightarrow\neg P(y,y))$ (is tautology)
C.6.13 There is no set containing all sets that do not contain themselves, i.e. $\{S\mid S\notin S\}$ is not a set.
C.6.14 The set $\{0,1\}^\infty$ is uncountable.
C.6.15 There are uncomputable functions $\mathbb{N}\to\{0,1\}$
C.6.16 The function $\mathbb{N}\to\{0,1\}$ assigning to each $y\in\mathbb{N}$ the complement of what program y outputs on input y, is uncomputable.