Logic Continued Continued

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Resolution Calculus - Clauses

Clause

A *clause* is a set of *literals*.

Here are some clauses:

$$\{A, B, C\}, \{A, \neg B\}, \varnothing$$

A clause is *satisfied* by a truth assignment if at least one of its literals is true. This means a clause "represents" a logical OR.

$$\{A, \neg B, C\} \sim (A \vee \neg B \vee C)$$

The empty clause is unsatisfiable. "One of the literals has to be true", but there are none.

Resolution Calculus - Sets of Clauses

Clause Set

A set $M = \{K_1, \ldots, K_m\}$ of clauses is satisfied by a truth assignment if **all** of its clauses are satisfied.

So a set of clauses "represents" the logical AND. For example

$$\{\{A, B\}, \{\neg A, C\}\}$$

Can be satisfied by setting $A = 1, B = 0, C = 1$.

The empty set of clauses is a tautology. "All have to be true", and none are all of none.

Resolution Calculus - Resolvents

Resolvent

Let $K_1 = \{A, L_1, \dots L_k\}$ and $K_2 = \{\neg A, l_1, \dots l_l\}$. Then the *resolvent* of K_1 and K_2 is $\{L_1, \dots, L_k, l_1, \dots, l_l\}$.

Resolution Calculus

The resolution calculus consists of one rule:

$$\{K_1, K_2\} \vdash_{res} K$$

if K is a resolvent of K_1, K_2. This rule is sound.

Resolution Calculus - Examples

Take:

$$L = \{A, B, C\}, M = \{\neg A, B\}, N = \{\neg C\}$$

Then we can resolve:

$$\{L, M\} \vdash_{res} \{C\}$$
$$\{L, N\} \vdash_{res} \{A, B\}$$
$$\{N, \{C\}\} \vdash_{res} \varnothing$$

Resolution Calculus - Usecase

Theorem 6.6

A set of formulas M is unsatisfiable exactly if $\mathcal{K}(M) \vdash_{Res} \emptyset$.

Here $\mathcal{K}(M)$ is simply the representation of M as a set of clauses.

Remember Lemma 6.3?

The following statements are equivalent:

1. $\{F_1, F_2, \ldots, F_k\} \models G$
2. $(F_1 \wedge F_2 \wedge \cdots \wedge F_k) \rightarrow G$ is a tautology
3. $\{F_1, F_2, \ldots, F_k, \neg G\}$ is unsatisfiable.

We can prove (3) using the resolution calculus! (And therefore also the other statements.)

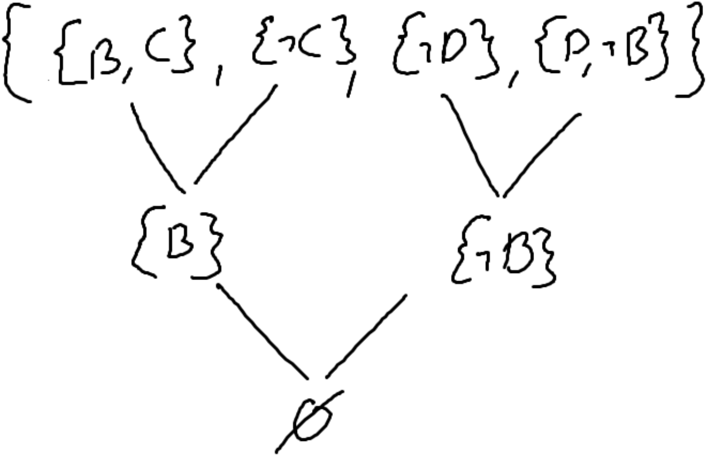
How this usually goes

1. They give you some statement / formula
2. You transform that statement / formula to a CNF
3. You use resolution calculus

Useful Tricks:

- Showing F is valid (tautology) is the same as showing $\neg F$ is unsatisfiable.
- Use Lemma 6.3
- Don't do a truth table if there are more than 3 variables! There is always some easy way (De-Morgan!?) to get to CNF!

Rare Proof-By-Picture Usecase



Exercise (Exam HS18)

($\star$) Let $F := ((A \to B) \to C) \vee (C \leftrightarrow \neg A)$. Find an equivalent formula in CNF (no justification is necessary). *(2 Points)*

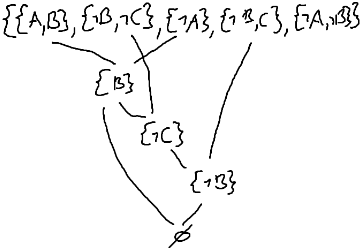
($\star$) Let $G := ((A \vee B) \to (B \wedge C)) \;\vee\; ((A \vee B) \wedge (A \vee \neg C)) \;\vee\; (A \wedge B)$. Using the resolution calculus, prove that G is **valid**. *(5 Points)*

https://exams.vis.ethz.ch/exams/d4dxjyo0.pdf

Solution

For the first one, the (minimal) solution is $A \vee C$, there are many others. You can find it quickly by doing a truth table.
For the second one we use De-Morgan and other rules to transform $\neg G$ into:

$$(A \vee B) \wedge (\neg B \vee \neg C) \wedge (\neg A) \wedge (\neg B \vee C) \wedge (\neg A \vee \neg B)$$



Since $\neg G$ is unsatisfiable, G is a tautology.

Rectified Form

Rectified Form

A formula is in *rectified form* if no variable is both unbounded and free and all bound variables are distinct.

We can easily transform a formula into rectified form by renaming the bound variables. For example:

$$\forall x \exists y (P(x) \wedge Q(y)) \wedge R(x, y)$$

becomes

$$\forall u \exists v (P(u) \wedge Q(v)) \wedge R(x, y)$$

You are only allowed to rename variables bound by a quantifier!

Prenex Form

Prenex Form

A formula is in *prenex form* if all the quantifiers are in front of one formula without quantifiers.

So for example, this formula is **not** in prenex form:

$$\forall x P(x) \land P(x)$$

But we can transform it to:

$$\forall u (P(u) \land P(x))$$

How To Prenex Form

1. Rename all variables bound by quantifiers
2. Use Lemma 6.7 to bring the quantifiers to the front

Lemma 6.7. *For any formulas F, G, and H, where x does not occur free in H, we have*

1) $\neg(\forall x\, F) \equiv \exists x\, \neg F$;
2) $\neg(\exists x\, F) \equiv \forall x\, \neg F$;
3) $(\forall x\, F) \land (\forall x\, G) \equiv \forall x\, (F \land G)$;
4) $(\exists x\, F) \lor (\exists x\, G) \equiv \exists x\, (F \lor G)$;
5) $\forall x\, \forall y\, F \equiv \forall y\, \forall x\, F$;
6) $\exists x\, \exists y\, F \equiv \exists y\, \exists x\, F$;
7) $(\forall x\, F) \land H \equiv \forall x\, (F \land H)$;
8) $(\forall x\, F) \lor H \equiv \forall x\, (F \lor H)$;
9) $(\exists x\, F) \land H \equiv \exists x\, (F \land H)$;
10) $(\exists x\, F) \lor H \equiv \exists x\, (F \lor H)$.

Exercise (Exam HS19)

($\star$) Find a formula in the prenex normal form which is equivalent to the folloing formula:
$\neg\forall x(P(x) \vee \neg Q(y)) \wedge \exists y(P(x) \vee Q(y))$. *(4 Points)*

Solution

$$
\begin{aligned}
&\neg\forall x(P(x) \vee \neg Q(y)) \wedge \exists y(P(x) \vee Q(y)) \\
&\equiv \neg\forall u(P(u) \vee \neg Q(y)) \wedge \exists v(P(x) \vee Q(v)) \\
&\equiv \exists u\neg(P(u) \vee \neg Q(y)) \wedge \exists v(P(x) \vee Q(v)) \\
&\equiv \exists u(\neg(P(u) \vee \neg Q(y)) \wedge \exists v(P(x) \vee Q(v))) \\
&\equiv \exists u(\exists v(P(x) \vee Q(v)) \wedge \neg(P(u) \vee \neg Q(y))) \\
&\equiv \exists u\exists v((P(x) \vee Q(v)) \wedge \neg(P(u) \vee \neg Q(y)))
\end{aligned}
$$