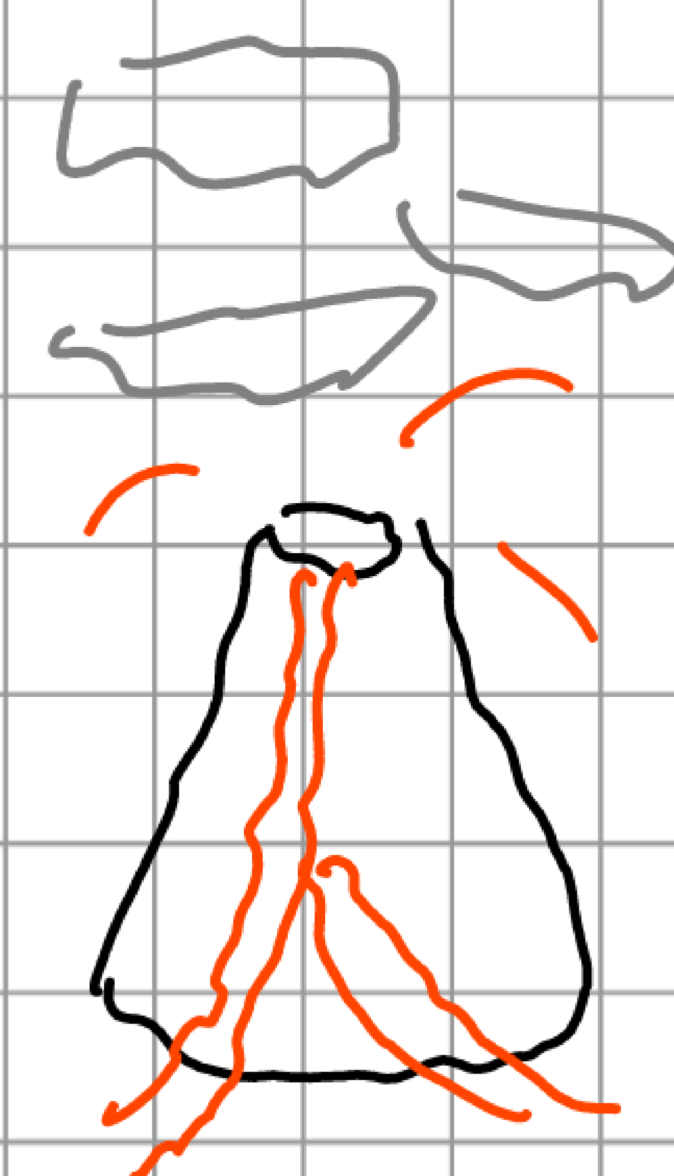
A Journey through Algebra

ALGEBRAS

An algebra is a carrier set A with some operations (functions). We write $\langle A; \circ \rangle$ for the algebra with the set A and operation $\circ$. This is nothing new, but just a categorisation of things we already know. For example $\langle \mathbb{N}; + \rangle$ is an algebra. They are also called "magmas", which is where everything begins...



SEMIGROUPS

You surely know from school that $(a+b)+c = a+(b+c)$. This is actually a very interesting property called associativity. If the operation of an algebra is associative we call it a semigroup.

Not relevant for this course.

An operation $*$ is associative if $(a*b)*c = a*(b*c)$

+	0	1
0	0	1
1	1	0

MONOIDS

Think about the natural numbers again. What is special about the number 0? Well, if we take any number and add 0 to it, it doesn't change the number. We call 0 a neutral element.

An algebra $\langle A; \circ, e \rangle$ with the neutral element e is called a monoid.

A left neutral element e_l has the property that $e_l * a = a \quad \forall a \in A$.
A right neutral element e_r is similar, just from the right: $a * e_r = a \quad \forall a \in A$.
A neutral element is both a right and left neutral element. We call it e: $a * e = e * a = a \quad \forall a \in A$.

Note that the operation of an algebra is not necessarily commutative. So $a * b = b * a$ does not need to hold.

GROUPS

Okay, we have monoids now. But wouldn't it be cool if we could somehow reverse the operation on an element? This is where inverses come into play. A monoid with an inverse function $\sim$ is denoted $\langle A; \circ, \sim, e \rangle$ and called a group.

A left/right inverse satisfies $\hat{a} * a = e \,/\, a * \hat{a} = e$.
An inverse is both a left and right inverse.