

Diskmath Week 6

Emil Straschil

A dark blue diagonal gradient bar that starts from the bottom left and extends towards the top right, covering the lower half of the slide.

Hello

I am Emil.

I have a **website**:

emils.site (yes thats the URL)

repo: <https://github.com/emil3tr/emil3tr.github.io>

All materials will be uploaded there.

Any Feedback / Questions /
Wishes / ... ?

→ estraschil@student.ethz.ch

→ “Emil” (floxi4) on dinfk-discord

→ Diskmath-questions: ask here
so others can benefit (:

Where are we right now?

Basics

Abstraction
Formulas
Statements
Prop. Logic
Pred. Logic
Proof Patterns

Sets and
Relations

Sets
Set Operations
Relations
Equivalence
Partial Order
Functions
Countability



Number Theory

Division
Primes
Modular Arith.
Diffie-Hellman

Algebra

Monoids
Groups
Euler Totient
RSA
Rings
Polynomials
Finite Fields
Err. Corr. Codes

Logic

Proof Systems
Logic
Calculi
Res. Calculus
Prop. Logic
Pred. Logic

Today

1. Chapter 3 - What's important?
2. Posets/Lattices/Functions - Learning by doing.
3. Countability Recap
4. Countability proofs

Questions?

Chapter 3 – What you should know

- Understand sets
- Set proofs (from set symbols to logic)
- Prove relation properties
- Know about equivalence / partial order relations
- Visualize relations with graphs
- Know edge cases to check:
 - empty set
 - empty relation
 - reflexivity
 - meet / join
 - ...
- Know what a function is
- Prove general function / relation properties (“take some element” proofs)

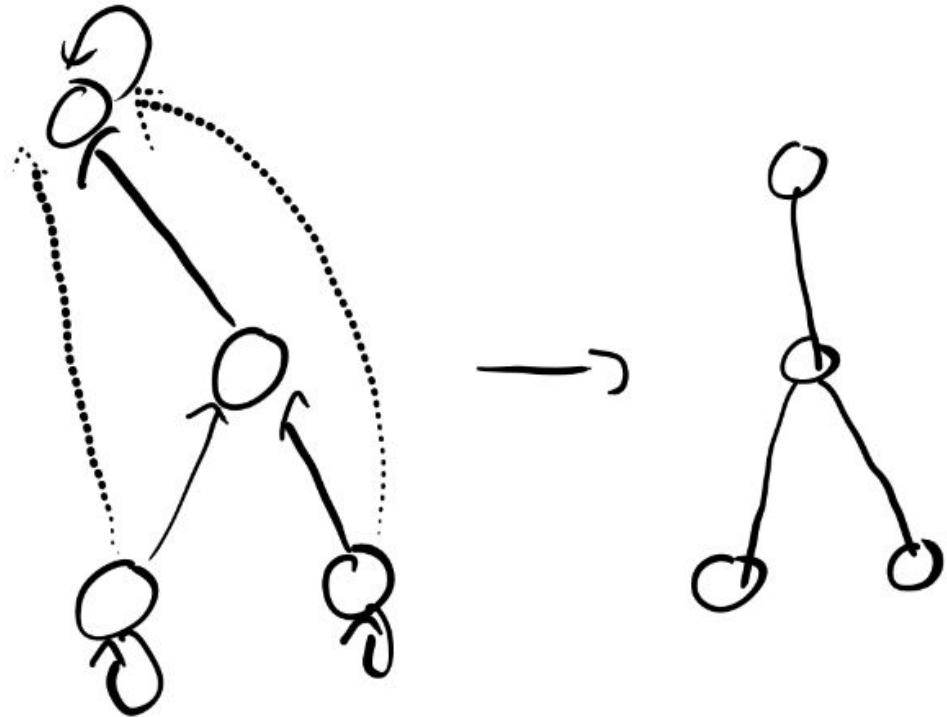
Posets

**reflexive, transitive,
antisymmetric**

Intuition: (a,b) means “a is ordered before b”

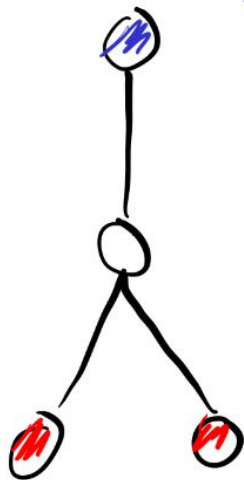
We draw **Hasse diagram**:

1. omit self loops
2. omit transitive edges
3. omit direction

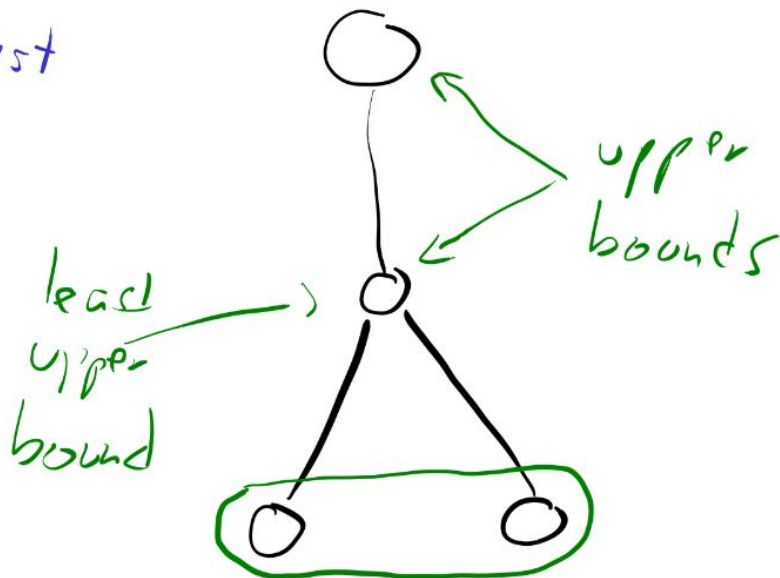


Special Elements

maximum/greatest



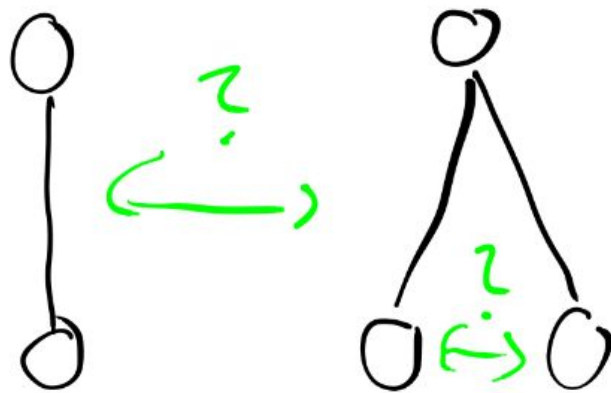
minimum
(no least)



Confusions

- There can be multiple minimal / maximal
- There can only be one least / greatest
- → There is either one least upper bound or none

- Some elements may simply be uncomparable



Lattice

We call the least upper bound
join.

We call the greatest lower bound
meet.

Every two elements have join and
meet = lattice

Kahoot

Break

Please fill out the feedback
form

Helps me A LOT

Fill out email from

(week 5 on emils.site)

Countability

Resources

discmath.ch guide by tobi is very good

guide on my website

Notation

$A < B$ means B dominates A

$A \sim B$ means $A < B$ and $B > A$

N are natural numbers

Q are rationals

R are reals

Countable

Intuition: we can enumerate the elements in the set

Formally: A is countable if

$$A \leq \mathbb{N}$$

so if there is an injection from A to $\mathbb{N}$

Countable sets are

- $\mathbb{N}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{Q}$
- finite bitstrings

Uncountable

Intuition: We cannot count the elements (they are too “dense”)

Formally: Not countable

Examples are

- infinite bitstrings

(99% of time we will use this for proofs)

(Also $\mathbb{R}$)

Important Sets

N x N is countable

finite bitstrings are countable

00101011010

and they have a length!

infinite bitstrings are countable

0101010110...

they have no length

Can use lemmas in script to make more countable sets.

Bitstring Tricks

Most exercises are solved with bitstrings.

A set of numbers can map to a bitstring!

$\{0, 2, 5, 6\}$ is 1010011

A infinite set can map to an infinite one

A function from $\mathbb{N}$ to $\{0,1\}$ is an infinite bitstring.

Position i has value $f(i)$.

$\{(0,1), (1,0), (2,1), \dots\}$ is 101...

Injection Intuition

If we have an injection from

A to B

then A is “smaller” than B.

(We can map each element from A a unique one from B.)

So $A < B$ is simply A “smaller” B.

So if A is uncountable and $A < B$ then B also uncountable.

If A is countable and $B < A$ then B also countable.

→ Proof strategy

Proving Countability

A set X is countable if $X < \mathbb{N}$.

But $\mathbb{N}$ is transitive and any countable set Y has $Y < \mathbb{N}$.

So if we show $X < Y$ then

$X < Y$ and $Y < \mathbb{N}$ and so

$X < \mathbb{N}$ and X countable

Strategy:

Want to show X countable

- 1) Find a known countable set Y
- 2) Find injection $X \rightarrow Y$
- 3) Make transitivity argument from left side

Proving Uncountability

Magic Sentence:

If Y is uncountable and we have
 $Y < X$ then X also uncountable.

Proof:

If X was countable
then $X < \mathbb{N}$
but then $Y < X$ and $X < \mathbb{N}$
and so $Y < \mathbb{N}$
but Y uncountable!

Proving Uncountability

Strategy:

Want to show X uncountable

- 1) Find uncountable set Y
- 2) Find injection $Y \rightarrow X$
- 3) Magic sentence

Most of the time use infinite bitstrings as set.

They are the only useful ones where we know from script that it is uncountable.

Defining a Function

Often times not as easy as

$$f(x) = 2 * x$$

Write something like:

“f is function from A to B and it maps a from A to b in B and b looks like ... “

Proving Injection $A \rightarrow B$

Need to show:

Well defined: Show that each element only maps to one.

Assume $x = y$ and show $f(x) = f(y)$
or assume $f(x) \neq f(y)$ and show
 $x \neq y$.

Totally defined: Show that each x in A has some $f(x)$ in B . Mostly obvious, but mention it.

Watch out for 0 when defining bitstrings!

Proving Injection $A \rightarrow B$

Injectivity:

Assume that $f(x) = f(y)$ and show that $x = y$.

Or assume $x \neq y$ and show $f(x) \neq f(y)$.

With bitstrings:

Often useful to assume that bitstrings are not equal. Then take the first bit where they differ and work with that one.

Sooo...

We see countability exercise, we do:

- 1) Intuition: Countable or not?
- 2) Find another set to work with
- 3) Find the injection
- 4) Prove that it is function
- 5) Prove that is is injection

6) Write down magic sentence 1 or 2

(from cheatsheet or brain)

Exercises